A STUDY OF SOME CLASSES OF TOPOLOGICAL VECTOR SPACES

GLEN SKANES

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BY

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Supervisor

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Internal Examiner

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External Examiner

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Date

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ABSTRACT

In this dissertation, we study two important classes of locally convex spaces in great detail. The first one is the class of (DF)-spaces, introduced by GROTHENDIECK as a prototype of duals of (F)-spaces. A non-locally convex analogue of these spaces is also discussed. The second one is the class of Schwartz spaces. The role of certain Banach spaces as "universal" to the "variety" of all Schwartz spaces is investigated. Finally, co-Schwartz spaces are introduced in a fashion analogous to the study of co-nuclear spaces from that of nuclear spaces. Several counter-examples are provided.

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CHAPTER I

DUALS AND BIDUALS OF (F)-SPACES

This chapter begins with a rapid survey of the duality theory of locally convex spaces to serve as a preparatory material for the subsequent chapters. Duality plays an important role in the development of (DF)- and Schwartz spaces, whose study is the major aim of this thesis. After recapitulating the standard notions and basic results, we deviate to the study of the strong duals and strong biduals of (F)-spaces. This is done to motivate the definition of a (DF)-space, introduced by GROTHENDIECK [7] as a prototype of strong duals of (F)-spaces. We conclude the chapter with a comprehensive, but not exhaustive treatment of important properties of these spaces.

Unless otherwise mentioned, we always follow the terminology of KOTHE [13]. Set theoretical difference is indicated by the symbol $\sim$, and $f|M$ denotes the restriction of the map f to the set M. We use the symbol # to indicate the end of a proof or of any numbered statement without proof.

1 Duality theory of locally convex spaces

All linear spaces considered are over the field of real or complex numbers, and all spaces considered are assumed to be Hausdorff. We denote a linear space E endowed with a vector topology τ by the symbol $E[\tau]$. The τ-*dual* of E is the space E' of all τ-continuous linear functionals on E. The *algebraic dual* E^* of E is the space of all linear functionals on E. The linear spaces E_1 and E_2 over the same field are said to be *in duality* if there exists a bilinear form $\langle\ ,\ \rangle$ on $E_1 \times E_2$ which separates the points of E_1, as well as E_2, in the sense of BOURBAKI [2].

We also say that $<E_1, E_2>$ is a dual pair. For such a pair, the *weak topology* $\tau_s(E_2)$ on E_1 is the weakest topology which renders each member of E_2 a continuous linear functional. Similarly $\tau_s(E_1)$ is the weak topology on E_2 defined by E_1.

Now, for a space $E[\tau]$, $<E,E'>$ is always a dual pair and the topology $\tau_s[E']$ on E is in general courser than τ. A topology on E is said to be *admissible for the pairing* $<E,E'>$ if the dual of E under the topology is precisely E'. Clearly the weak topology is the weakest admissible topology for $<E,E'>$. A comprehensive method for defining various topologies on E arising from the pairing $<E,E'>$ is by means of polars. For each set $A \subset E$, the *polar* $A^o = \{f \in E' : |<x,f>| \leq 1$ for each x in $A\}$ and is an absolutely convex $\tau_s(E)$-closed subset of E'. Dually for $B \subset E'$, $B^o = \{x \in E : |<x,f>| \leq 1$ for each f in $B\}$. The bipolar theorem asserts that A^{oo} is the $\tau_s(E)$-closed absolutely convex hull of A. The symbol $\Gamma(A)$ refers to the absolutely convex hull of A. Corresponding to each family $\mathcal{M}$ of $\tau_s(E)$-bounded sets of E', the family $\{p_M : M \in \mathcal{M}\}$ of seminorms defined by $p_M(x) = \sup \{|<x,f>| : f \in M\}$ prescribes a locally convex topology on E. The family $\mathcal{M}$ is said to be *saturated* if

1) $M \in \mathcal{M}$, $N \subset M$ implies $N \in \mathcal{M}$,

2) $M \in \mathcal{M}$ implies $\lambda M \in \mathcal{M}$ for each scalar λ, and

3) $M, N \in \mathcal{M}$ implies that the $\tau_s(E)$-closed absolutely convex hull of $M \cup N$ belongs to $\mathcal{M}$.

If $\mathcal{M}$ is *total* in E' in the sense that $\cup\{M : M \in \mathcal{M}\}$ spans E', then the generated topology on E is finer than the weak topology $\tau_s(E')$. If $\mathcal{M}$ is saturated, the converse is true.

If M is any family of $\tau_s(E)$-bounded sets, there is a smallest saturated family M^* which contains M. If M is total and saturated, the family $\{M^o : M \in M\}$ of polars of convex $\tau_s(E)$-closed members of M forms a neighbourhood basis for the polar topology. This topology is usually referred to as the *topology of uniform convergence on the members of* M (or simply the *M-topology*). The weak topology $\tau_s(E')$ in E is obtained by choosing M to consist of all finite subsets of E'. The *initial topology* τ is realized by setting M to be the saturated family of all τ-equicontinuous subsets of E'. The *Mackey topology* $\tau_K(E')$ and the *strong topology* $\tau_b(E')$ on E are respectively the M-topologies corresponding to the families of all absolutely convex $\tau_s(E)$-compact sets, and $\tau_s(E)$-bounded sets of E'.

That $\tau_K(E')$ is the largest admissible topology for $<E,E'>$ is the content of the Mackey-Arens Theorem. Yet another topology, $\tau_{b^*}(E')$ on E is obtained by choosing M to be the saturation of all $\tau_b(E)$-bounded subsets of E'. Clearly, $\tau_s(E') < \tau < \tau_K(E') < \tau_{b^*}(E') < \tau_b(E')$ when "<" refers to "coarser than". In addition to these we also need a topology denoted by $\tau_c(E)$ namely the topology of uniform convergence on the precompact subsets of $E[\tau]$. This is a topology on E'.

The space $E[\tau]$ is said to be *barrelled* if $\tau = \tau_b(E')$ and *infrabarrelled* if $\tau = \tau_{b^*}(E')$. Equivalently $E[\tau]$ is barrelled if and only if each *barrel* (i.e. closed absolutely convex absorbing set) is a τ-neighbourhood of 0 ; and $E[\tau]$ is infrabarrelled if each *bornivorous* (a set which absorbs all bounded sets) barrel is a τ-neighbourhood of O. A *bornological space* is a locally convex space in which every absolutely convex bornivorous set is a neighbourhood of O. The *bornological topology* τ^* associated with τ is the finest locally convex topology whose family of bounded sets is identical with the family of τ-bounded sets. Clearly $E[\tau]$ is bornological if and only if $\tau = \tau^*$.

The bidual E'' of $E[\tau]$ is the $\tau_b(E)$-dual of E'. The canonical map $I : E \to E''$ defined by $I(x)(f) = \langle x,f\rangle$ for x in E, f in E' imbeds E into the bidual E''. The space $E[\tau]$ is *semi-reflexive* if the map I is onto, or equivalently on E', the topologies $\tau_K(E)$ and $\tau_b(E)$ coincide. Now $\langle E',E''\rangle$ forms a dual pair. The strong bidual of $E[\tau]$ is the space $E''[\tau_b(E')]$ arising from the pairing $\langle E',E''\rangle$. The space $E[\tau]$ is *reflexive* if and only if the canonical map I of E onto the strong bidual is an isomorphism onto. It is well-known that a space is reflexive if and only if it is both semi-reflexive and infrabarrelled.

We shall also consider the *natural topology* $\tau_n(E')$ on E'' which is the topology of uniform convergence on the τ-equicontinuous subsets of E'. We remark that while $\tau_n(E')$ induces the initial topology τ on E, the topology $\tau_b(E')$ on E'' induces $\tau_{b*}(E')$ on E. We say $E[\tau]$ is *distinguished* if its strong dual is barrelled, or equivalently $\tau_b(E) = \tau_b(E'')$ on E'.

For a subspace H of $E[\tau]$ we have a pairing $\langle H,E'/H^\perp\rangle$. The topology $\tau_s(E')$ relativized to H coincides with $\tau_s(E'/H^\perp)$, while the relativizations of $\tau_K(E')$ and $\tau_b(E')$ are coarser than $\tau_K(E'/H^\perp)$ and $\tau_b(E'/H^\perp)$, respectively. If H is a closed linear subspace of E and τ_q represents the quotient topology on E/H, the dual of $E/H[\tau_q]$ can be identified with $H^\perp$ and $\tau_s(H^\perp)$ on E/H is precisely the quotient topology $\tau_s(E')_q$.

For a family $\{E_\alpha : \alpha \in A\}$ of locally convex spaces ΠE_α denotes the *product* space with the product topology. The *algebraic direct sum* $\bigoplus E_\alpha$ is the subspace of ΠE_α consisting of those members with only a finite number of non-zero co-ordinates. The *locally convex direct sum topology* on $\bigoplus E_\alpha$ is the finest locally convex topology rendering each injection $I_\alpha : E_\alpha \to \bigoplus E_\alpha$ continuous. A neighbourhood basis for this topology consists of sets of the form $\Gamma_\alpha I_\alpha(U^\alpha_\beta)$ where for a fixed α, $\{U^\alpha_\beta\}$ represents a neighbourhood basis of 0 in E_α and $\Gamma_\alpha I_\alpha(U^\alpha_\beta)$ denotes the absolutely convex hull of the union of the $I_\alpha(U^\alpha_\beta)$.

More generally, let $\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ denote a family of locally convex spaces, and for each α, let $u_\alpha : E_\alpha \to E$ denote a linear map, where E is the span of the set $\bigcup_{\alpha\in A} u_\alpha(E_\alpha)$. Let τ denote the finest locally convex (not necessarily Hausdorff) topology on E which renders each u_α continuous. If τ is Hausdorff, then $E[\tau]$ is called the *inductive limit* (or *locally convex hull*) of the family $\{E_\alpha[\tau_\alpha], u_\alpha\}_{\alpha\in A}$. We write this by the statement $E[\tau] = \sum_{\alpha\in A} u_\alpha[E_\alpha[\tau_\alpha]]$. If A is a countable set, then $E[\tau]$ is called the *generalized strict inductive limit* of the family. In addition, if each u_n is the identity map and E is the union of the increasing sequence of subspaces $\{E_n\}$, then the generalized strict inductive limit topology induces on each E_n a topology coarser than τ_n. If for each n, τ_{n+1} restricted to E_n is the topology τ_n, then $E[\tau]$ is called the *strict inductive limit* of the family. An absolutely convex absorbing set V of E is a neighbourhood of 0 for the strict inductive limit topology if and only if $V \cap E_n$ is a τ_n-neighbourhood of 0 in E_n for each n.

Let $f : E \to F$ be a linear map between two locally convex spaces. The linear map $f^* : F^* \to E^*$ defined by $f^*(y^*)(x) = y^*(f(x))$ for x in E and y^* in F^* is called the *algebraic adjoint* of f. If f is continuous, then it is $\tau_s(E') - \tau_s(F')$ (weakly) continuous, and consequently, $f^*(F') \subset E'$. Hence, if f is weakly continuous we can define the *transpose* *mapping* $f' : F' \to E'$ by $f'(y')(x) = y'(f(x))$ for x in E and y' in F'.

In this case f' is weakly continuous, and, further, f' is also $\tau_b(F) - \tau_b(E)$ continuous. Again, its transpose $f'' : E'' \to F''$ is well-defined and f'' restricted to E coincides with $f : E \to F$.

2 Strong duals and strong biduals of (F)-spaces

A complete metrizable locally convex space is an (F)-space. It forms an important class of topological vector spaces, and occurs naturally in the theory of distributions and many other situations in analysis. An (F)-space is always barrelled, Baire, and bornological and possesses many pleasant properties such as the Krein-Smulian Property for convex sets [9], and satisfies the open mapping and closed graph theorems [9]. It is well-known that the weak dual of an (F)-space possesses a number of striking properties. In contrast with this and in contrast with the strong duals of Banach spaces, the strong dual E' of an (F)-space E has a structure more complicated than that of the space E itself. For example, $E'[\tau_b(E)]$ is not metrizable unless E is normable; also, $E'[\tau_b(E)]$ need not be barrelled, and hence neither bornological nor infrabarrelled.

Nevertheless, the fact that $E'[\tau_b(E)]$ possesses a fundamental sequence of bounded sets has far-reaching consequences, and this will be developed in what follows. This discussion motivates the study of a class of spaces called (DF)-spaces, introduced first by GROTHENDIECK [7] as a natural generalization of the strong duals of (F)-spaces.

1.1. DEFINITION. A locally convex space $E[\tau]$ is said to possess a *fundamental sequence of bounded sets* if there exists an increasing sequence $\{B_n\}$

of bounded sets in E such that every bounded set of E is contained in some B_n. #

Clearly such a fundamental sequence exists in every Banach space. We now investigate under what circumstances such a sequence exists in an (F)-space. We start with the following result.

1.2 THEOREM If $E[\tau]$ is locally convex, metrizable and possesses a fundamental sequence $\{B_n\}$ of bounded sets, then it is normable.

PROOF We first claim that there is a bounded set B in E which absorbs every other bounded set. If not let $B_0 = \{0\}$ and we may assume that each B_n is absolutely convex and does not absorb B_{n+1}.

Let $x \in B_1 \sim \{0\}$ and set $x_n = x/n$. Obviously $x_n \in B_1$ for each n and (x_n) converges to 0. For each pair (n,k) of integers ≥ 1, choose $z_{n,k} \in \frac{1}{k} B_n \sim (k+1)B_{n-1}$ with $z_{n,k} \neq -x_n$. Let $M = \{x_n + z_{n,k}: n,k = 1,2\ldots\}$. For a fixed n, consider the sequence $\{z_{n,k}\}_{k=1}^{\infty}$. Since B_n is bounded, for an absolutely convex neighbourhood U of 0, there exists an integer $m > 0$ with $B_n \subset mU$. Consequently, for $k \geq m$, $z_{n,k} \in \frac{1}{k} B_n \subset \frac{m}{k} U \subset U$ and hence $z_{n,k} \to 0$. Hence $x_n + z_{n,k} \to x_n \in \bar{M}$ and $0 \in (\bar{M})^{-} = \bar{M}$. Again no Cauchy sequence in M has 0 as a limit. For each term of such a Cauchy sequence is of the form $x_n + z_{n,k}$; and the indices n occuring in the terms form an unbounded set. Otherwise, some x_{n_o} repeats infinitely often, and the subsequence $\{x_{n_o} + z_{n_o,k}\}$ converges to 0. But $z_{n_o,k} \to 0$ and so $z_{n_o,k} = -x_{n_o}$ for some k, which is impossible. Since the sequence is bounded, there exists a fixed integer m, with $x_n + z_{n,k} \in B_m$. Hence $z_{n,k} \in B_m + B_1 \subset 2B_m$. By the choice of $z_{n,k}$ we then must have $B_{n-1} \subsetneq B_m$, so $n \leq m$ a contradiction. Observe that the closure of a set in a metric space is determined by its Cauchy sequences.

Since B absorbs each B_n, B^o is absorbed by each B_n^o in $E'[\tau_b(E)]$. Thus E' is normable with closed unit ball B^o. Therefore, the strong dual $E''(\tau_b(E'))$ is also normable with B^{oo} as the closed unit ball. But $\tau_b(E')$ on E'' induces $\tau_{b*}(E')$ on E and because $E[\tau]$ is metrizable and hence infrabarrelled, we obtain that $E[\tau]$ is normable. #

As immediate consequences of the above theorem we have the following.

1.3 COROLLARY The strong dual of a locally convex metrizable space, possessing a fundamental sequence of bounded sets, is normable. #

1.4 COROLLARY An (F)-space is a Banach space if and only if it has a fundamental sequence of bounded sets. #

Next, if $E[\tau]$ is locally convex and metrizable, and $\{U_n\}$ is a decreasing sequence of neighbourhoods, then since E is infrabarrelled, each strongly bounded subset M of E' is contained in an equicontinuous set U^o where U is a neighbourhood of 0 in E and $U \supset U_n$ for some n; hence $M \subset U^o \subset U_n^o$. Consequently $\{U_n^o\}$ forms a fundamental sequence of strongly bounded subsets of E'. This observation together with corollary 1.3 yields the following theorem.

1.5 THEOREM Let $E[\tau]$ be locally convex and metrizable. Then $E'[\tau_b(E)]$ is metrizable if and only if $E[\tau]$ is normable. #

To complete the necessity part of the proof we have only to observe that $E''[\tau_b(E')]$ is normable and since $\tau_b(E')$ induces τ on E, $E[\tau]$ is normable. The sufficiency is trivial.

A second criterion for an (F)-space to be a Banach space is provided below.

1.6 THEOREM An (F)-space E is a Banach space if and only if E contains a bounded absorbent set.

PROOF The closed unit ball is the required set for the necessity part and for the sufficiency, if B is bounded and absorbent, the closed absolutely convex hull C of B is a barrel, and is a neighbourhood of 0. Consequently, E is normable. #

Let us recall that if $\{U_n\}$ is a decreasing sequence of absolutely convex sets forming a neighbourhood basis of 0 in a locally convex metrizable space $E[\tau]$, then the sequence $\{U_n^{oo}\}$ of polars in E'' forms a neighbourhood basis for the natural topology $\tau_n(E')$ on E''. Hence clearly $E''[\tau_n(E')]$ is metrizable. Since $E[\tau]$ is infrabarrelled, the strongly bounded subsets of E' are equicontinuous. Noting that the weakly bounded subsets of E' in the pairing $\langle E',E''\rangle$ are the same as the above-mentioned strongly bounded subsets of E' we get the following observation.

1.7 REMARK If $E[\tau]$ is locally convex and metrizable, then $\tau_n(E') = \tau_b(E')$ on E''. #

The next lemma will be crucial in realizing the structure of the strong duals of (F)-spaces. The conclusions will play a prominent role in defining the class of (DF)-spaces in the next chapter.

1.8 LEMMA Suppose $E[\tau]$ is locally convex and metrizable with strong dual $E'[\tau_b(E)]$ and strong bidual $E''[\tau_n(E')]$. If $\{M_n\}$ is a sequence of $\tau_b(E)$-equicontinuous subsets of E'' whose union M is $\tau_s(E')$-bounded in E'', then M is $t_b(E)$-equicontinuous.

PROOF Let $\{U_n\}$ be as in the remarks above with the additional assumption that each U_n is closed. Since M_n is equicontinuous, $M_n \subset B_n^{oo}$, where B_n is an absolutely convex bounded subset of $E[\tau]$, and B_n^{oo} is the polar in E''.

By KÖTHE [13; 21,6.(4)] $E'[\tau_b]$ is complete. Again, by the Banach-Mackey Theorem [13; 20,11.(8)] M is $\tau_n(E')$ bounded. Hence for each τ_n-neighbourhood U_k^{oo} of 0, there is a constant $c_k > 0$ with $M \subset c_k U_k^{oo}$. Further, for each k and n, there exists $a_{nk} > 0$ with $B_n \subset a_{nk} U_k$. Set $b_k = \max\limits_{n \leq k} (c_k, a_{nk})$. Then $\bigcap\limits_{k=n+1}^{\infty} a_{nk} U_k \subset \bigcap\limits_{n+1}^{\infty} b_k U_k$ so

$(\bigcap\limits_{n+1}^{\infty} b_k U_k)^o \subset (\bigcap\limits_{n+1}^{\infty} a_{nk} U_k)^o \subset B_n^o$. Therefore we have since $B_n^{oo} \supset M_n$, that

$(\bigcap\limits_{n+1}^{\infty} b_k U_k)^o \subset M_n^o$.

Since $M_n \subset M \subset b_k U_k^{oo}$ it follows that

$\frac{1}{b_k} U_k^o \subset M_n^o$, so $M_n^o \supset (\overline{\Gamma_{k=1}^{n} \frac{1}{b_k} U_k^o}) = (\bigcap\limits_{k=1}^{n} b_k U_k)^o$ by [13; 20,6.(5) and 20,8.(10)] where the closure refers to the $\tau_s(E)$-closure. Hence,

$2M_n^o = M_n^o + M_n^o \supset (\bigcap\limits_{k=1}^{n} b_k U_k)^o + (\bigcap\limits_{k=n+1}^{\infty} b_k U_k)^o$. Now $(\bigcap\limits_{k=1}^{n} b_k U_k)^o$ being the polar of a neighbourhood of 0 in $E[\tau]$ is weakly compact in E'. Thus

$(\bigcap\limits_{k=1}^{n} b_k U_k)^o + (\bigcap\limits_{k=n+1}^{\infty} b_k U_k)^o$ is weakly closed. Consequently

$(\bigcap\limits_{k=1}^{\infty} b_k U_k)^o \subset (\bigcap\limits_{k=1}^{n} b_k U_k)^o + (\bigcap\limits_{n+1}^{\infty} b_k U_k)^o \subset 2M_n^o$. Hence, for all n,

$\frac{1}{2} M_n \subset (\bigcap\limits_{k=1}^{\infty} b_k U_k)^{oo}$ and so $\frac{1}{2} M \subset (\bigcap\limits_{k=1}^{\infty} b_k U_k)^{oo}$. Since $\bigcap\limits_{k=1}^{\infty} b_k U_k$ is bounded in E, the desired conclusion follows. #

The next theorem gives an interesting property possessed by the strong biduals of metrizable spaces.

1.9 THEOREM If $E[\tau]$ is locally convex and metrizable, its strong bidual $E''[\tau_n(E')]$ is a sequentially $\tau_s(E')$-complete (F)-space.

PROOF If $\{U_n\}$ is a decreasing sequence of neighbourhoods of 0 which forms a basis, the sequence $\{U_n^{oo}\}$ forms a basis of neighbourhoods of 0 for $\tau_n(E')$ on E''.

Let $\{x_n\}$ be a Cauchy sequence in $E''[\tau_s(E')]$. Then $\{x_n\}$ is $\tau_s(E')$-bounded and hence $\tau_b(E)$-equicontinuous. Consequently $\{x_n\}$ is relatively $\tau_s(E')$-compact with a compact $\tau_s(E')$-closure X. Thus $\{x_n\}$ converges to some point of X, in the topology $\tau_s(E')$ on E''. So $E''[\tau_s(E')]$ is sequentially complete. Again $E''[\tau_n(E')]$ has a basis of neighbourhoods of 0 consisting of $\tau_s(E')$-closed sets and $\tau_n(E')$ is finer than $\tau_s(E')$. Hence by [13; 18,4.(4)] $E''[\tau_n(E')]$ is sequentially complete. Consequently $E''[\tau_n(E')]$ is an (F)-space, being metrizable. #

As a special case of the above theorem we obtain

1.10 COROLLARY The strong bidual of an (F)-space is an (F)-space. #

We conclude this chapter with the following two results which are extensions of theorems valid for Banach spaces.

1.11 THEOREM If $E[\tau]$ is a non-reflexive (F)-space, the strong bidual $E''[\tau_n(E')]$ is a non-reflexive (F)-space. #

1.12 COROLLARY If E is a non-reflexive (F)-space, the iterated strong duals are all non-reflexive, and in each of the sequences

$$E \subset E'' \subset E'''' \subset \ldots \quad \text{and} \quad E' \subset E''' \subset \ldots$$

each space is a proper subspace of its successor. #

CHAPTER II

(DF)-SPACES

In this chapter we attempt a systematic treatment of a class of spaces known as (DF)-spaces first introduced by GROTHENDIECK [7]. The motivation for the study of these spaces is contained in the statement of lemma 1.8 regarding the strong duals of locally convex metrizable spaces. The class of (DF)-spaces plays an important role in the theory of topological tensor products and nuclear spaces. This class contains the class of all normed spaces and the class of all infrabarrelled spaces which possess a fundamental sequence of bounded sets. While the strong dual of every (F)-space is a (DF)-space, the converse is not true.

As an important property of these spaces we show that the topology of a (DF)-space can be "localized" in a fashion analogous to the localization of the topology of compact convergence on the dual of an (F)-space. While most of the stability properties are available for the class of (DF)-spaces we provide an example to show that a closed subspace of a (DF)-space need not be a (DF)-space. We obtain conditions under which the property of being a (DF)-space is preserved while passing to closed subspaces.

The final section of the chapter is devoted to the study of bornological (DF)-spaces which form an important class of (DF)-spaces in their own right. Examples are given to show that an infrabarrelled (DF)-space need not be bornological.

3 <u>Basic properties of (DF)-spaces</u>

2.1 DEFINITION A locally convex space $E[\tau]$ is said to be a *(DF)-space* [7] if

a) it has a fundamental sequence of bounded sets and

b) every strongly bounded subset of E' which is a countable union of τ-equicontinuous sets is τ-equicontinuous.

We can replace the condition (b) by the following equivalent condition:

b') if $\{U_n\}$ is a sequence of closed absolutely convex neighbourhoods of 0 in $E[\tau]$, and if $U = \bigcap_{n=1}^{\infty} U_n$ absorbs every bounded set, then U is a τ-neighbourhood of 0. #

An infrabarrelled locally convex space with a fundamental sequence of bounded sets is always a (DF)- space; hence every normed space is a (DF)-space. It is clear from lemma 1.8 that the strong dual of a metrizable space is a complete (DF)-space.

An example of a complete (DF)-space which is not topologically isomorphic to the strong dual of a metrizable space is provided later [cf. II.4].

The first result of the section, due to GROTHENDIECK [7], describes the neighbourhood system at 0 for a (DF)-space.

2.2 THEOREM Let $E[\tau]$ be a (DF)-space and let $\{B_n\}$ be a fundamental sequence of closed absolutely convex bounded subsets of E. Then an absolutely convex subset W, of E is a τ-neighbourhood of 0 if and only if $W \cap B_n$ is a τ-neighbourhood of 0 in B_n for each n.

PROOF We need only to prove the sufficiency part. We construct a sequence $\{\alpha_n\}$ of positive numbers and a sequence $\{U_n\}$ of closed absolutely convex neighbourhoods of 0 such that

a) $\alpha_n B_n \subset \frac{1}{3} W$

b) $\alpha_n B_n \subset U_k$, and

c) $U_n \cap B_n \subset W$, for all n and for all k.

If $m = 1$, choose U_1' to be an absolutely convex and closed neighbourhood of 0 such that $U_1' \cap B_1 \subset W \cap B_1 \subset W$. Let $0 < \alpha_1 \leq \frac{1}{3}$ so that $\alpha_1 B_1 \subset \frac{1}{3} U_1'$. Then $\alpha_1 B_1 \subset \frac{1}{3} U_1' \cap \frac{1}{3} B_1 \subset \frac{1}{3} W$. Thus we can set $U_1 = \frac{1}{3} U_1'$.

Suppose for each $n \leq m$, α_n and U_n have been chosen so as to satisfy conditions (a), (b), and (c). By assumption, there is a neighbourhood U of 0 with $U \cap B_{m+1} \subset W$. Choose $0 < \alpha_{m+1} \leq \frac{1}{3}$ so that $\alpha_{m+1} B_{m+1} \subset \frac{1}{3} U$; then $\alpha_{m+1} B_{m+1} \subset \frac{1}{3} W$ and so (a) is satisfied. Clearly α_{m+1} can be chosen small enough to admit (b) for $n = m + 1$ and $k \leq m$.

Let $B^{(m+1)} = \overline{\Gamma_{i=1}^{m+1} \alpha_i B_i}$. If we find an absolutely convex neighbourhood V of 0 such that $U_{m+1} = \overline{B^{(m+1)} + V}$ satisfies (c) for $n = m + 1$, then since $\alpha_n B_n \subset B^{(m+1)} \subset U_{m+1}$, (b) is satisfied for $n \leq m + 1$ and $k = m + 1$. Now $B^{(m+1)} + V$ is an absolutely convex neighbourhood of 0 and so $\overline{B^{(m+1)} + V} \subset 2B^{(m+1)} + 2V$; hence in order to prove (c), it suffices to show that $(2B^{(m+1)} + 2V) \cap B_{m+1} \subset W$, or equivalently, denoting $B_{m+1} \cap (E \sim W)$ by M, that $(2B^{(m+1)} + 2V) \cap M = \phi$ for a suitable neighbourhood V. Thus it amounts to showing that $2V \cap (M + 2B^{(m+1)}) = \phi$ or, setting $N = M + 2B^{(m+1)}$, is suffices to show that $0 \notin \bar{N}$.

As $B^{(m+1)} \subset \frac{1}{3} W$, we have $\frac{1}{3} W + 2B^{(m+1)} \subset W$, hence $\frac{1}{3} W \cap N = \phi$ because $W \cap M = \phi$. Clearly N and therefore $3N$ is bounded, and so $3N \subset B_k$ for some k. Then $W \cap B_k$ is a neighbourhood of 0 in B_k and $W \cap B_k \cap 3N = \phi$. Hence $0 \notin \overline{3N}$, and so $0 \notin \bar{N}$. Thus our construction is valid for all n and k.

To complete the proof note that $U = \bigcap_{k=1}^{\infty} U_k$ absorbs every bounded set by (b) and again by 2.1(b') U is a τ-neighbourhood of 0. Since $E = \bigcup_{n=1}^{\infty} B_n$, (c) is used to obtain that $U \subset W$. Hence W is a τ-neighbourhood of 0. #

We have the following simple corollary.

2.3 COROLLARY Let $E[\tau]$ be locally convex and metrizable and let $\{U_n\}$ be a basis of absolutely convex neighbourhoods of 0 which form a decreasing sequence. An absolutely convex set W of E' is a $\tau_b(E)$-neighbourhood of 0 in E' if and only if $W \cap U_n^o$ is a $\tau_b(E)$-neighbourhood of 0 in U_n^o for each n. #

If $\{B_n\}$ is any fundamental sequence of bounded sets in a (DF)-space $E[\tau]$, then $\{\overline{\Gamma B_n}\}$ forms a fundamental sequence of absolutely convex closed bounded sets in E. If further $f : E \to F$ is a linear map, from E into the locally convex space $F[\tau']$, whose restriction to each B_n is continuous then it is easily seen that the restrictions of f to each $\overline{\Gamma B_n}$ is also continuous, since for each n, there is a k with $\overline{\Gamma B_n} \subset B_k$. Again if $V \subset F$ is an absolutely convex neighbourhood of 0, then $f^{-1}[V] \cap \overline{\Gamma B_n}$ is a neighbourhood of 0 in $\overline{\Gamma B_n}$ because of the continuity of the restrictions. So by theorem 2.2, $f^{-1}[V]$ is a neighbourhood of 0 in E. Thus we have the following result.

2.4 COROLLARY A linear map f from a (DF)-space E into a locally convex space F is continuous if and only if its restriction to each member of a fundamental sequence of bounded sets is continuous. #

Similar to the result that the strong dual of an (F)-space is a (DF)-space, we now have the following pleasant situation, namely

2.5 THEOREM If $E[\tau]$ is a (DF)-space, then $E'[\tau_b(E)]$ is an (F)-space.

PROOF The metrizability of $E'[\tau_b(E)]$ follows because $E[\tau]$ has a fundamental sequence of bounded sets. Since the bounded sets of $E[\tau]$ are saturated and total, by the completeness theorem of GROTHENDIECK [16; Ch. 6, Sec. 1, Th. 2] a linear map on E is in the completion of E' if and only if its restriction to each bounded set of E is continuous. Hence by 2.4 $E'[\tau_b(E)]$ is complete. #

The next result, again due to GROTHENDIECK [7] relates the topology of a (DF)-space to that of the topology $\tau_{b*}(E')$ on E.

2.6 THEOREM If M is a separable subset of a (DF)- space $E[\tau]$, the topologies τ and $\tau_{b*}(E')$ coincide on M.

PROOF Since $\tau_{b*}(E')$ is finer than τ and $x + M$ is a separable subset of $E[\tau]$ if and only if M is a separable subset , we need only to show that given an absolutely convex closed $\tau_{b*}(E')$-neighbourhood V of 0, there exists an open τ-neighbourhood U of 0 with $M \cap U \subset V$. (Without loss of generality we may assume that $0 \in M$.) We can assume that V is τ-closed, for a basis of neighbourhoods of 0 for $\tau_{b*}(E')$ is given by the polars of the strongly bounded subsets of E'. Hence we have to establish that $(M \cap U) \cap (E \sim V) = \phi$.

Let $\{x_i\}$ be a dense sequence in M so that $M \subset \overline{\{x_i\}}$. Because $U \cap (E \sim V)$ is open, it suffices to show that no x_i is a member of $U \cap (E \sim V)$; namely, there is a U which contains none of the x_{i_n}'s which lie in $E \sim V$. Rename these x_{i_n} again by $x_1, x_2, \ldots,$ If these x_n's are finite the result is immediate since E is Hausdorff. Hence we assume that $\{x_n\}$ forms a sequence.

We now construct sequences of numbers $\alpha_n > 0$ and closed absolutely convex neighbourhoods U_n of 0 satisfying

a) $\alpha_n B_n \subset U_k$

b) $\alpha_n B_n \subset V$ and

c) $x_n \notin U_n$, for each n and k.

For $m = 1$, the construction is trivial since τ and $\tau_{b^*}(E')$ have the same bounded sets. Suppose for $n,k \leq m$ the desired sequences have been constructed. Then it is possible to choose α_{m+1} so that (b) is satisfied, and (a) is satisfied for $n = m + 1$ and $k \leq m$. Let $B^{(m+1)} = \sum_{n=1}^{m+1} \alpha_n B_n$. Since $B^{(m+1)} \subset V$ and $x_{m+1} \notin V$, by [13; 15,6(9)], there exists an absolutely convex neighbourhood U^* of 0 in $E[\tau]$ such that $(x_{m+1} + U^*) \cap (B^{(m+1)} + U^*) = \phi$. Letting $U_{m+1} = B^{(m+1)} + U^*$, (c) is satisfied and also (a) for $n \leq m + 1$ and $k = m + 1$.

Finally putting $U' = \bigcap_{n=1}^{\infty} U_n$, by (a), U' absorbs every bounded set and so by the definition of a (DF)-space U' is a neighbourhood of 0. Choosing a τ-open $U \subset U'$, the proof is completed since by (c) $x_n \notin U_n$ for any n. #

The conclusion of the above theorem yields two sufficient conditions for a (DF)-space to be infrabarrelled. More precisely we have

2.7 THEOREM

a) Every separable (DF)-space is infrabarrelled.

b) If the bounded subsets of a (DF)-space $E[\tau]$ are metrizable, then $E[\tau]$ is infrabarrelled.

PROOF Statement (a) is immediate from 2.6. To prove (b) in view of 2.2, we shall show that τ and $\tau_{b^*}(E')$ coincide on every bounded set B. Since in each B, τ is metrizable, it is determined by the class of sequentially closed subsets of B. While every $\tau_{b*}(E')$-convergent net is τ-convergent, by 2.6 it is straightforward to see that every τ-convergent sequence in B is $\tau_{b*}(E')$-convergent in B.

The fact that τ is metrizable in B shows that the τ-closure of a set in B is contained in the $\tau_{b*}(E')$-closure of the set. Hence every $\tau_{b*}(E')$-closed set in B is τ-closed in B. The reverse implication is trivial and hence the restriction of these topologies to B coincide. #

Let us recall that a locally convex space $E[\tau]$ is distinguished if its strong dual $E'[\tau_b(E)]$ is barrelled. The previous result immediately gives a criterion for an (F)-space to be distinguished. The fact that the strong dual of an (F)-space is a complete (DF)-space together with theorem 2.7 (b) yields the following result.

2.8 COROLLARY An (F)-space is distinguished if the bounded subsets of the strong dual are metrizable. #

4 Example of a (DF)-space which is not the strong dual of an (F)-space

We now give an example of a complete (DF)-space which is not topologically isomorphic to the strong dual of a metrizable space. Further, the space we construct is not even infrabarrelled.

Let B denote a non-separable reflexive Banach space with the strong (= norm) dual B'. Let τ be the topology of uniform convergence on the separable bounded subsets of B' (where B' has the norm topology). Since singleton sets of B' are bounded and separable, the topology τ is admissible for the pairing $\langle B,B'\rangle$. Consider the class M of all bounded, separable subsets of the Banach space B'. Separability of a set is still retained by passing to countable unions or taking closures. If X is a separable subset of a topological vector space with a countable dense subset $\{x_n\}$, set $Z = \bigcup_{n=1}^{\infty} \{ \sum_{j=1}^{n} \alpha_j x_j \}$ where each α_j is rational if the field is $\mathbb{R}$ and has rational real and imaginary parts if the field is $\mathbb{C}$ and $\sum_{j=1}^{n} |\alpha_j| \leq 1$. Clearly Z is a countable subset of ΓX and $\Gamma X \subset \bar{Z}$. Hence ΓX is separable.

One can easily check that M is saturated. Hence M is the class of all τ-equicontinuous subsets of B'. Since B is not separable, the same is true for B'. Consequently the closed unit ball of B' is not separable and hence does not belong to M. Thus τ is strictly coarser than the Mackey topology $\tau_K(B')$ since a Banach space is barrelled.

If $B[\tau]$ is topologically isomorphic to the strong dual of a metrizable space, then the strong bidual of this metrizable space is topologically isomorphic to B' which implies that the strong bidual is normable and complete. Hence $B[\tau]$ would have to be a Banach space which is false since $\tau \neq \tau_K(B')$.

We show that $B[\tau]$ is complete. By [13; 21,10.(3)] every precompact set of B' lies in the closed absolutely convex hull of a null sequence in B'. Obviously every null sequence is bounded and separable in B' and hence is contained in M. Since M is saturated, every precompact subset of B' is in M. Hence, τ_c, the topology of uniform convergence on the precompact subsets of B' is coarser than τ.

By [13; 21,6.(4)] since B' is metrizable, $B[\tau_c(B')]$ is complete. Again by [13; 18,4.(4)] since $B[\tau]$ has a basis of neighbourhoods of 0 consisting of τ_c-closed sets $B[\tau]$ is complete.

Finally $B[\tau]$ is a (DF)-space because it has the same bounded sets as the Banach space B and again a strongly bounded set which is a countable union of sets in M is in M.

5. Hereditary properties of (DF)-spaces

We now study some permanence properties possessed by (DF)-spaces. While many of the standard properties carry over, the main drawback is that a subspace of a (DF)-space need not be a (DF)-space even when it is closed. This situation will be investigated in greater detail and various conditions are obtained under which this can be answered in the affirmative. First we study the obvious properties.

2.9 THEOREM If H is a closed subspace of a (DF)-space $E[\tau]$ then $E/H[\tau_q]$ is also a (DF)-space and on $H^\perp$ the topologies $\tau_b(E)|H^\perp$ and $\tau_b(E/H)$ coincide.

PROOF Let $^\circ$ and * denote polars in the pairings $\langle E,E'\rangle$ and $\langle E/H,H^\perp\rangle$ respectively, and K denote the canonical surjective map of E onto E/H. A basis of neighbourhoods of 0 for $\tau_b(E)|H^\perp$ is given by the family of $B^\circ \cap H^\perp$ where B is $\tau_s(E')$-bounded in E. Since B is τ-bounded as well, $K[B]$ is τ_q-bounded in E/H and $(K[B])^* = B^\circ \cap H^\perp$. Hence, $\tau_b(E/H)$ is finer than $\tau_b(E)$ restricted to $H^\perp$. To show the reverse inequality, we show that the identity map

$$I : H^\perp[\tau_b(E)|H^\perp] \to H^\perp[\tau_b(E/H)] \quad \text{is}$$

continuous. Since $H^\perp[\tau_b(E)|H^\perp]$ is metrizable, it is a bornological space, and so by [13; 28,3.(4)], I is continuous if and only if every sequence $\{u_n\} \subset H^\perp$ which is $\tau_b(E)$-convergent to 0 is $\tau_b(E/H)$-bounded. Now $E[\tau]$

being a (DF)-space, $\{u_n\}$ is τ-equicontinuous, hence relatively $\tau_s(E)$-compact, and therefore relatively $(\tau_s(E)|H^\perp)$-compact since $H^\perp$ is $\tau_s(E)$-closed. By [13; 22,2.(2)] $\{u_n\}$ is relatively $\tau_s(E/H)$-compact and so is $\tau_b(E/H)$-bounded. Thus the second assertion is proved.

We prove that $E/H[\tau_q]$ is a (DF)-space by first exhibiting a fundamental sequence of bounded sets satisfying (a) of definition 2.1. If A is a bounded set in E/H and $\{B_n\}$ is a fundamental sequence of bounded sets in $E[\tau]$, there exists a B_n with $B_n^o \cap H^\perp \subset A^*$, so $(K[B_n])^* \subset A^*$ and hence $A \subset (K[B_n])^{**}$. Consequently $E/H[\tau]$ has a fundamental sequence of bounded sets. The verification of (b) of 2.1 is trivial. #

As a sort of dual result we have

2.10 THEOREM If the subspace $H[\tau_H]$ of a locally convex space $E[\tau]$ is a (DF)-space, the strong dual of $H[\tau_H]$ is topologically isomorphic to $E'/H^\perp[\tau_b(E)_q]$.

PROOF We have to show that the algebraic isomorphism of $H'[\tau_b(H)]$ into $E'/H^\perp$ is topological. Using an argument similar to 2.9, it is easily seen that $\tau_b(H)$ is coarser than $\tau_b(E)_q$. Now $H'[\tau_b(H)]$ is metrizable, so by [13; 28,3.(4)], $\tau_b(H)$ is finer than $\tau_b(E)_q$, if every sequence $\{\hat{u}_n\}$ convergent to 0 in $H'[\tau_b(H)]$ is $\tau_b(E)_q$-bounded. Since $H[\tau_H]$ is a (DF)-space, $\{\hat{u}_n\}$ is τ_H-equicontinuous in H'. By [13;22,1.(1)] $\{\hat{u}_n\}$ is the canonical image of a τ-equicontinuous subset of E'. Since the latter set is $\tau_b(E)$-bounded, its canonical image $\{\hat{u}_n\}$ is $\tau_b(E)_q$-bounded in $E'/H^\perp$. #

The next result deals with the completeness properties of a (DF)-space.

2.11. THEOREM

a) A (DF)-space $E[\tau]$ is complete if and only if it is quasi-complete.

b) The completion of a (DF)-space is again a (DF)-space.

c) Every semi-reflexive (DF)-space is complete.

PROOF In 2.10 take $E[\tau] = H[\tau]$ a (DF)-space which is a subspace of its completion $\tilde{E}[\tilde{\tau}]$. Because $E^{\perp} = \{0\}$ the strong topologies $\tau_b(E)$ and $\tau_b(\tilde{E})$ coincide on $E' = (\tilde{E})'$ which may be regarded as equal since E is dense in $\tilde{E}[\tilde{\tau}]$. We take polars in the pairing $\langle\tilde{E},E'\rangle$. If A is bounded in $\tilde{E}$, then there is a $B \subset E$ such that B is absolutely convex, τ-closed, and τ-bounded and $B^{o} \subset A^{o}$. It follows that $A \subset \tilde{B} = B^{oo}$ where the closure is taken in $\tilde{E}[\tilde{\tau}]$. If $x \in \tilde{E}$, then $x \in \tilde{B}$ where B is τ-closed and τ-bounded in E. But since $E[\tau]$ is quasi-complete, B is complete under τ, and hence complete under $\tilde{\tau}$; hence $B = \tilde{B}$ and $x \in E$. Because $E = \tilde{E}$, the proof of (a) is immediate.

By our remarks above, if A is bounded in $\tilde{E}[\tilde{\tau}]$, and $\{B_n\}$ is a fundamental sequence of bounded sets in $E[\tau]$, then there exists n such that $A \subset \tilde{B}_n$ where the closure is taken in $\tilde{E}[\tilde{\tau}]$. Hence, $\tilde{E}[\tilde{\tau}]$ has a fundamental sequence of bounded sets. Let $\{U_n\}$ be a sequence of closed absolutely convex neighbourhoods of 0 in $\tilde{E}[\tilde{\tau}]$ for which $U = \bigcap_{n=1}^{\infty} U_n$ absorbs all bounded sets. It follows that $U \cap E = \bigcap_{n=1}^{\infty} U_n \cap E$ is bornivorous in $E[\tau]$, so $U \cap E$ is a τ-neighbourhood of 0 in E. Again $\widetilde{U \cap E}$ is a $\tilde{\tau}$-neighbourhood of 0 in $\tilde{E}$, and since $\widetilde{U \cap E} \subset U$, the proof of (b) is completed.

To prove (c) it need only be noted that every semi-reflexive space is quasi-complete by [17; Ch. IV, Cor. 1 to Th. 5.5]. #

Let $\{E_n[\tau_n]\}_{n=1}^{\infty}$ be a sequence of (DF)-spaces and $F = \bigoplus_{n=1}^{\infty} E_n[\tau_n]$ be the locally convex direct sum of the sequence. By [13; 18,5.(4)] every bounded set B in F is contained in some set of the form $\bigoplus_{i=1}^{N} B_{n_i}$, where B_{n_i} is bounded in $E_{n_i}[\tau_{n_i}]$ for $i = 1, \ldots, N$. So if $\{B_n^{(m)}\}_{m=1}^{\infty}$ is a fundamental sequence of bounded sets in $E_n[\tau_n]$, then the sets $B_m = \bigoplus_{n=1}^{m} B_n^{(m)}$, $m = 1,2, \ldots,$ form a fundamental sequence of bounded sets in F.

Let $U = \bigcap_{m=1}^{\infty} U_m$ where each U_m is a closed absolutely convex neighbourhood of 0 in F, and U absorbs every bounded set in F. It follows easily from the definition of the direct sum topology that U is a neighbourhood of 0 in F.

Thus we have proved that F is a (DF)-space. By [13; 19,1.(3)] every topological inductive limit (= locally convex hull) of the spaces $E_n[\tau_n]$ is topologically isomorphic to a quotient space of F by a closed linear subspace H. The following theorem is then immediate.

2.12 THEOREM The locally convex hull $F[\tau] = \sum A_n[E_n[\tau_n]]$ of a sequence of (DF)-spaces $E_n[\tau_n]$ is again a (DF)-space. #

The next result identifies each bornological (DF)-space as a topological inductive limit of normed spaces.

2.13 THEOREM A locally convex space $F[\nu]$ is a bornological (DF)-space if and only if it is the topological inductive limit of an increasing sequence of normed spaces.

PROOF The sufficiency is easy to prove by 2.12 and [13; 28,4.(1)]. If $\{B_n\}$

is a fundamental sequence of closed bounded absolutely convex sets, it can be shown that $E[\tau]$ is the topological inductive limit of the spaces E_n by the injective mappings $I_n : E_n \to E$, where E_n is the linear subspace of E generated by B_n, with a norm topology which has B_n as the closed unit ball. Hence, the necessity. #

We now return to the discussion of the locally convex direct sum $F = \bigoplus_{n=1}^{\infty} E_n[\tau_n]$ of a sequence of (DF)-spaces $\{E_n[\tau_n]\}_{n=1}^{\infty}$. It follows from the proof of 2.9, that each bounded set of the quotient F/H of F by a closed linear subspace H is contained in the closure of the canonical image of some bounded set in F. Every bounded set in F is contained in some $\bigoplus_{i=1}^{N} B_{n_i}$; hence there exists some m and bounded absolutely convex sets $B_n \subset E_n[\tau_n]$ for which $\Gamma_{n=1}^{m} B_n$ contains the bounded set (where each B_n is considered as its image in F). Let the linear map $A : F \to \sum A_n[E_n[\tau_n]]$ be defined by $A(\bigoplus_1^{\infty} x_n) = \sum A_n(x_n)$; then A is continuous and open, and it can be shown that every bounded subset of the locally convex hull is contained in some set of the form $\Gamma_{n=1}^{m} A_n[B_n]$ where B_n is bounded in $E_n[\tau_n]$.

2.14 THEOREM The locally convex hull of a sequence of semi-reflexive (reflexive) (DF)-spaces $\{E_n[\tau_n]\}_{n=1}^{\infty}$ is again a semi-reflexive (reflexive) (DF)-space.

PROOF In both cases, the $E_n[\tau_n]$ are semi-reflexive; since B_n is τ_n-bounded, therefore B_n is relatively $\tau_s(E_n')$- compact because $\tau_b(E_n)$ is admissible for the pairing $\langle E_n', E_n\rangle$. Let $E = \sum A_n[E_n[\tau_n]]$ be a locally convex hull for the sequence of spaces. Each A_n is continuous, hence $\tau_s(E_n') - \tau_s(E')$ continuous. By [13; 20,6.(5)], the absolutely convex cover of finitely many $A_n[B_n]$ is relatively $\tau_s[E']$-compact, consequently, with the remarks preceeding this theorem, we have E is semi-reflexive.

If the $E_n[\tau_n]$ are reflexive, they are barrelled. By [13; 27,1.(3)], $E = \sum A_\pi[E_n[\tau_n]]$ is barrelled; and a barrelled semi-reflexive space is reflexive. #

6 Example of a closed subspace of a (DF)-space which is not a (DF)-space

We need a few notions from the theory of perfect sequence spaces. For a vector sequence space λ (which is always assumed to be a linear subspace of the space ω of all scalar sequences), the *Köthe dual* λ^* is defined to be the space of all scalar sequences $\{u_n\}$ such that $\sum_{n=1}^{\infty} |u_n x_n| < \infty$ for each $\{x_n\}$ in λ. Clearly $\lambda \subset \lambda^{**}$ and we say λ is *perfect* if $\lambda = \lambda^{**}$. If $\{\lambda_\alpha\}$ is a class of sequence spaces, the linear hull $\sum_\alpha \lambda_\alpha$ is again a sequence space, and if each λ_α is perfect, so is the sequence space $\bigcap_\alpha \lambda_\alpha$. Also $(\sum_\alpha \lambda_\alpha)^* = \bigcap_\alpha \lambda_\alpha^*$ if each $\lambda_\alpha \supset \phi$ (the space of all finite sequences). If further each λ_α is perfect, $(\bigcap_\alpha \lambda_\alpha)^* = (\sum_\alpha \lambda_\alpha^*)^{**}$. If $a = \{a_n\}$ is an arbitrary vector, we denote by λ_a the perfect space of all $\{x_n\} \in \omega$ with $\sum_{n=1}^{\infty} |a_n|\,|x_n| < \infty$.

Let $a^{(k)}$, $k = 1,2,\ldots,$ be a countable number of vectors in ω, called *steps*. If the vectors are co-ordinate-wise increasing over k, then the system of steps is said to be *monotonic increasing*. Then $\bigcap_{k=1}^{\infty} \lambda_{a^{(k)}}$ and $\sum_{k=1}^{\infty} \lambda^*_{a^{(k)}}$ are respectively called the *echelon space* and *co-echelon space* corresponding to the $a^{(k)}$. A detailed account of such spaces, is given in KÖTHE [13; 30].

An infrabarrelled space is called a *Montel space ((M)-space)* if every bounded set is relatively compact. Further, an (M)-space is said to be an

(FM)-space (*(DFM)-space*) provided it is also an (F)-space ((DF)-space).

We start with a sequence of vectors written as double sequences. For each $k = 1,2,3, \ldots$ take

$$a^{(k)} = (a_{11}^{(k)}, a_{12}^{(k)}, \ldots; a_{21}^{(k)}, a_{22}^{(k)}, \ldots; \ldots; \ldots)$$

$$= (b_1^{(k)}; \ldots; b_{k-1}^{(k)}; k^k e; k^{k+1} e; \ldots)$$

with $b_j^{(k)} = (1, 2^k, 3^k, \ldots)$ and $e = (1, 1, \ldots)$.

Denote by λ the echelon space corresponding to the $a^{(k)}$, and let λ^* be its Köthe dual. We can make the sequence of vectors $a^{(k)}$ monotonic by making the addition $a'^{(k)} = a^{(1)} + \ldots + a^{(k)}$. Obviously the echelon space for the new sequence of vectors remains the same as the original one. It can be shown that the conditions of [13; 30,9.(1)] are satisfied, so that $\lambda[\tau_K(\lambda^*)]$ is an (FM)-space and $\lambda^*[\tau_K(\lambda)]$ is an (M)-space. Further, when $k = 1$, then $a^{(1)}$ has each component $a_{ij}^{(1)} = 1$; so $\lambda_{a^{(1)}} = \ell^1$ and $\lambda^*_{a^{(1)}} = \ell^\infty$; i.e. $\lambda^*_{a^{(1)}}$ is ℓ^∞ written as a space of double sequences.

Let A be a linear map on λ defined by $A(x_{ij}) = (\sum_{i=1}^\infty x_{i1}, \sum_{i=1}^\infty x_{i2}, \ldots)$. Since $a^{(1)} \in \lambda^*$ and $\sum_{j=1}^\infty |\sum_{i=1}^\infty x_{ij}| \leq \sum\sum |x_{ij}| < \infty$ we have $A(x_{ij}) \in \ell^1$. If $u = (u_j) \in \ell^\infty$ and $x = (x_{ij}) \in \lambda$, we have $u(A(x_{ij})) = \sum_{j=1} u_j \sum_{i=1} x_{ij} = (u; u; \ldots)x$, so that the transpose $A' : \ell^\infty \to \lambda^*$ is well-defined since λ^* contains the double-sequence representation of ℓ^∞ and $A'(u) = (u; u; \ldots)$. It is easy to see that A' is a one-to-one weakly continuous map from ℓ^∞ into λ^* with a weakly closed image space. By BOURBAKI [2; Vol.2, p. 106], it follows that A is open, onto, and continuous as a mapping of (F)-spaces. If N is the kernel of A, then the space λ/N with the topology $\tau_K(\lambda^*)_q$ is isomorphic to ℓ^1. So the (FM)-space $\lambda[\tau_K(\lambda^*)]$ has a quotient space which is not an (M)-space.

Since λ is an (M)-space, the closure of the canonical image of every bounded set is compact; consequently, since λ/N is not an (M)-space, there is a bounded set in λ/N not contained in the closure of the canonical image of a bounded set in λ. Using lemma 4.4, we easily see that the strong topology $\tau_b(\lambda/N)$ is strictly finer than the topology $\tau_b(\lambda)$ restricted to $H = N^{\perp} \subset \lambda^*$.

Because $\lambda^*[\tau_b(\lambda)]$ is a complete (DF)-space, $H[\tau_b(\lambda)|H]$ is a complete space, and from the earlier remarks H is not barrelled, and hence, not infrabarrelled. Using [13; 21,3.(4) and 27,2.(3)], it can be shown that $H[\tau_b(\lambda)|H]$ is separable. By theorem 2.6, we thus have a closed subspace of a (DF)-space which is not a (DF)-space.

Next we prove that the property of being a (DF)-space is hereditary on passing to a subspace of finite co-dimension; and if the initial space is sequentially complete or barrelled, then on passing to a subspace of countable co-dimension. These results are due to VALDIVIA [21].

2.15 LEMMA Let G be a subspace of co-dimension one of a locally convex space E. If U is an absolutely convex closed bornivorous set of G, such that its closure $\bar{U}$ in E is absorbing, then $\bar{U}$ is bornivorous in E.

PROOF Let $\mathcal{B}$ denote the family of all absolutely convex closed and bounded sets B in E. We denote by E_B, as usual, the linear hull $\bigcup_{n=1}^{\infty} nB$ of B, with the norm topology with B as its closed unit ball. Let $E[\tau] = \sum_B E_B$ be the locally convex hull of the spaces $\{E_B\}_{B\in\mathcal{B}}$ by the injection mappings. Since τ is finer than the initial topology on E, the locally convex hull is Hausdorff. Because the E_B are bornological, so is $E[\tau]$ by [13; 28,4.(1)]. By a result of DIEUDONNÉ [4], since G is of finite co-dimension, $G[\tau']$ is bornological if $\tau' = \tau|G$. If U' is a neighbourhood of 0 in $E[\tau]$, then $U' \cap E_B$ is a neighbourhood of 0 in E_B, and so there exists $\lambda > 0$ with $B \subset \lambda(U' \cap E_B)$. Consequently $B \subset \lambda U'$. Obviously $E[\tau]$ and E have the same bounded sets; hence $E[\tau]$ and $G[\tau']$ are the bornological spaces associated with E and its subspace G.

Let $\bar{U}$ and $\bar{U}^*$ be the closures of U in E and $E[\tau]$. If G is dense in $E[\tau]$, then $\bar{U}^*$ is a neighbourhood of 0 in $E[\tau]$. So we have $\bar{U}$ is a bornivorous set in E. If not, then G is closed in $E[\tau]$. Let L be the subspace generated by $x \in \bar{U} \sim U$ (and hence $x \notin G$) and let L have the induced Hausdorff topology. Then $G + L = E$ and $G \cap L = \{0\}$. Let p and $q = I - p$ be the projections $p : E[\tau] \to G[\tau']$ and $q : E[\tau] \to L$. By [9; Ch.2, Prop. 7.1 and 10.4], the maps p and q are continuous. Let C be the absolutely convex hull of $\{x\}$.

If $B \in \mathcal{B}$, there is an $r > 0$ with $p[B] \subset rU$, and $q[B] \subset rC$; consequently $B \subset p[B] + q[B] \subset r(U + C)$. So $U + C$ is a neighbourhood of 0 in $E[\tau]$. Now $2\bar{U} = \bar{U} + \bar{U} \supset U' + C$; hence $2\bar{U}$ and, thus $\bar{U}$, are bornivorous in E. #

2.16 THEOREM Let E be a (DF)-space. If G is a subspace of E of finite co-dimension, then G is a (DF)-space.

PROOF Let $\{x_1, x_2, \ldots, x_n\}$ be a co-base of G. Then considering respectively the linear hulls of $G \cup \{x_1, \ldots, x_n\}$, $G \cup \{x_1, \ldots, x_{n-1}\}, \ldots,$ $G \cup \{x_1\}$, G, it suffices to carry out the proof where G is of co-dimension one ; so we shall make this assumption. Two cases are possible.

1) G *is closed in* E. If x is an element of E, but not G, let L be the linear hull of $\{x\}$. Then $G + L = E$, and $G \cap L = \{0\}$. By ROBERTSON and ROBERTSON [16; Ch.V, Prop.29 and Corollary], G and E/L

are topologically isomorphic. Using 2.9, $G^{\wedge}$ is a (DF)-space.

2) G *is dense in* E. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of absolutely convex closed neighbourhoods of 0 in G such that $U = \bigcap_{n=1}^{\infty} U_n$ absorbs all bounded sets of G. Let $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded absolutely convex closed sets in E; then $\{B_n \cap G\}_{n=1}^{\infty}$ is a fundamental sequence of absolutely convex bounded closed sets in G. Let f be a linear functional associated with the hyperspace $G \subset E$. Then if $f^{-1}[\{0\}] \cap B_n = G \cap B_n$ is $\tau_s(E')$-closed for each n, we have $f \in E'$ by ROBERTSON and ROBERTSON [16; Ch.VI, Prop. 1.2], since $E'[\tau_b(E)]$ is complete. This contradicts the fact that G is dense in E. Hence for some m, $B_m \cap G$ is not closed in E. Choose x in $\overline{(B_m \cap G)} \sim (B_m \cap G)$; then $\overline{B_m \cap G} \subset \overline{B_m} = B_m$ which verifies $x \notin G$. Because U is bornivorous in G, there exists $\lambda > 0$ with $B_m \cap G \subset \lambda U$; hence $\overline{B_m \cap G} \subset \lambda \bar{U}$ and $x \in \lambda \bar{U}$, so that $\frac{1}{\lambda}x \in \bar{U}$. Any $y \in E$ can be written as $y = g + \alpha x$ for some $g \in G$ and scalar α. We can easily show $\bar{U}$ is absorbent. By the lemma 2.15 $\bar{U}$ is bornivorous in E.

For each n, $\bar{U} \subset \overline{U_n}$, thus $\bigcap_{n=1}^{\infty} \overline{U_n}$ is bornivorous in E. Since E is a (DF)-space, $\bigcap_{n=1}^{\infty} \overline{U_n}$ is a neighbourhood of 0 in E. But $[\bigcap_{n=1}^{\infty} \overline{U_n}] \cap G = \bigcap_{n=1}^{\infty} (\overline{U_n} \cap G) = \bigcap_{n=1}^{\infty} U_n = U$ is a neighbourhood of 0 in G.

We now prove a series of lemmas which will be needed to prove the major results.

2.16 LEMMA Let F and G be two subspaces of a locally convex space E, such that $F \subset G$, and the co-dimension of F in G is finite. Let $\mathcal{B}$ be the family of all bounded, closed and absolutely convex sets of E. If for each $B \in \mathcal{B}$, $F \cap B$ is closed, then $G \cap B$ is closed.

PROOF Obviously, it suffices to carry out the proof where F is of co-dimension one in G.

It is first shown that $F \cap E_B$ is closed in E_B, for each $B \in \mathcal{B}$. Let a sequence $\{x_n\}_{n=1}^{\infty} \subset F \cap E_B$ converge to $x \in E_B$. Since the sequence is bounded, there exists $\lambda > 0$ with $\lambda x_n \in B$, $n = 1,2, \ldots$. Now $F \cap B$ is closed in E, and the initial topology restricted to E_B is coarser than the norm topology on E_B, hence $F \cap B$ is closed in E_B. Thus, since λx_n converges to λx, we have $\lambda x \in F \cap B$, and $x \in F \cap E_B$.

Let τ be the topology on G where $G[\tau]$ is the locally convex hull of the spaces $\{E_B \cap G\}_{B \in \mathcal{B}}$ by the injection mappings into G; this topology is finer than the initial topology restricted to G. Since F is a hyper-plane of $G[\tau]$, and for each $B \in \mathcal{B}$, $F \cap E_B \cap G$ is closed in $E_B \cap G$, we have F is closed in $G[\tau]$ by [13; 19,1.(7)].

Let $x \in G$ with $x \notin F$. Then $G[\tau]$ is the locally convex direct sum of $F[\tau|F]$ and the linear hull L of $\{x\}$ with the Hausdorff vector topology. Let I be the identity on G, and p and $q = I - p$ be the projections of G onto L and F. Let $\tau|F = \tau'$. Then p and q are continuous with respect to the topologies τ and τ'.

Given $B \in \mathcal{B}$, we now show $B \cap G$ is closed in E. We take a net $\{x_\alpha\}_{\alpha \in A} \subset B \cap G$ which converges to $x \in E$. To prove the result we need to show that $x \in B \cap G$. For each α, $x_\alpha = p(x_\alpha) + q(x_\alpha)$. Since $B \cap G$ is bounded in $G[\tau]$, $p[B \cap G]$ is bounded, so relatively compact, in L. There is a subnet $\{y_\gamma\}_{\gamma \in \Gamma}$ of $\{x_\alpha\}_{\alpha \in A}$ such that $p(y_\gamma) \to y \in L$. Thus, $p(y_\gamma) \to y$ in the initial topology on E, and $\lim_{\gamma \in \Gamma} q(y_\gamma) = \lim_{\gamma \in \Gamma}(y_\gamma - p(y_\gamma)) =$ $= x - y$ in E.

As $q[B \cap G]$ is bounded in $F[\tau']$ and τ' is finer than the induced topology from E, there exists $B' \in \mathcal{B}$ with $q[B \cap G] \subset B' \cap F$. Hence

$q(y_\gamma) \in B' \cap F$ for each γ and since $F \cap B'$ is closed, $x - y = z \in B' \cap F \subset F$. So $x = y + z$ where $y \in L$ and $z \in F$ implies $x \in G$, and B is closed implies $x \in B$, i.e. $x \in B \cap G$. #

The next lemma gives a condition under which a (DF)-space is a strict inductive limit of a sequence of its subspaces.

2.18 LEMMA Let E be a (DF)-space. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of subspaces of E, such that $\bigcup_{n=1}^{\infty} E_n = E$. If for each bounded set A, there exists a positive integer m, with $A \subset E_m$, then E is the strict inductive limit of the sequence $\{E_n\}_{n=1}^{\infty}$.

PROOF The locally convex hull $E[\tau]$ of the spaces E_n is the strict inductive limit of these spaces by [9; Ch.2, Sec.12, Th.1 and Cor.1]. Suppose U is an absolutely convex τ-neighbourhood of 0 in E; then $U \cap E_n$ is a neighbourhood of 0 in E_n for each n. If A is bounded and absolutely convex, there exists a positive integer m with $A \subset E_m$. Consequently $U \cap A = (U \cap E_m) \cap A$ is a neighbourhood of 0 in A. By 2.2, U is a neighbourhood of 0 in E. #

2.19 LEMMA Let E be a (DF)-space and let $\mathcal{B}$ be the family of all absolutely convex bounded closed subsets of E. Let G be a subspace of infinite countable co-dimension in E. If, for each $B \in \mathcal{B}$, $G \cap B$ is closed and G is of finite co-dimension in the linear hull of $G \cup B$, then G is closed.

PROOF We show that if $x \in E \sim G$, there is an open $A \subset E$ with $A \cap G = \phi$ and $x \in A$. Let $\{x, x_1, \ldots\}$ be a co-base of G in E and let H be the hyperplane generated by $G \cup \{x_1, x_2, \ldots\}$. By hypothesis, G is of finite co-dimension in the linear hull of $G \cup B$ given $B \in \mathcal{B}$, so there is an integer m such that B is contained in the linear hull E_m of $G \cup \{x, x_1, \ldots, x_m\}$.

If H_m is the linear hull of $G \cup \{x_1, \ldots, x_m\}$ we have $H \cap B = H \cap (E_m \cap B) = (H \cap E_m) \cap B = H_m \cap B$. Since the co-dimension of G in H_m is finite, we have $H_m \cap B$ is closed by lemma 2.17. Hence the hyperplane H intersects each $B \in \mathcal{B}$ in a closed set. As in the proof of theorem 2.16, H is closed in E. Choose A to be the complement of H. #

Now we are ready to prove the main result.

2.20 THEOREM Let E be a (DF)-space. Let G be a subspace of E. If for every bounded set B of E, G is of finite co-dimension in the linear hull of $G \cup B$, then G is a (DF)-space.

PROOF Let $\{B_n\}$ be a fundamental sequence of bounded sets in E. For each n, let F_n be a co-base of G in the linear hull of $G \cup B_n$ such that for each n, $F_n \subset F_{n+1}$. Since each F_n is finite, the set $F = \bigcup_{n=1}^{\infty} F_n$ is a countable co-base of G in E. If the co-dimension of G in E is finite, the result holds by theorem 2.16. If not, we let $\{x_1, x_2, \ldots\}$ be a co-base of G, and set $E_1 = G$, and for each $n \geq 2$, set E_n to be the linear hull of $G \cup \{x_1, \ldots, x_{n-1}\}$. The proof is carried out in three cases.

1) G *is closed in* E. According to lemma 2.18, E is the strict inductive limit of the sequence $\{E_n\}_{n=1}^{\infty}$. Let H be the subspace generated by $\{x_1, x_2, \ldots\}$ and p be the projection of E onto E_1 with kernel H. If p_n is the restriction of p to E_n, for $n = 1, 2, \ldots$, then p_n is continuous because E_1 is closed and of finite co-dimension in E_n. By [13; 19,1.(7)] p is continuous. So as in theorem 2.16, $G = E_1$ is topologically isomorphic to the (DF)-space E/H.

2) G *is dense in* E. Since E has a fundamental sequence of bounded sets, so does G; hence we need only show that if $\{U_n\}_{n=1}^{\infty}$ is a sequence of closed absolutely convex neighbourhoods of 0 in G, and $U = \bigcap_{n=1}^{\infty} U_n$ absorbs each bounded set in G, then U is a neighbourhood of 0 in G.

Let $\bar{U}_n$ and $\bar{U}$ be the closures in E of U_n and U, for $n = 1,2, \ldots,$ and let L be the linear hull of $\bar{U}$. We shall show that $E = L$ and $\bar{U}$ is bornivorous in E. Let B be an arbitrary closed absolutely convex bounded set in L, and E_B the linear hull of B with the gauge of B in L as norm. Note that $G + E_B$ is the linear hull of $G \cup B$. If $G \cap E_B$ has infinite co-dimension in E_B, there exists an infinite-dimensional subspace C of E_B, with $G \cap E_B + C = E_B$, $G \cap E_B \cap C = \{0\}$. Hence, $G + C \subset G + E_B$, which is a contradiction; so, $G \cap E_B$ has finite co-dimension in E_B. We consider $G \cap E_B$ to be a topological subspace of the normed space E_B. Since $B \cap G$ is bounded in G, there exists $\lambda > 0$ with $B \cap G \subset \lambda U$; consequently, $U \cap [G \cap E_B] = U_1$ is bornivorous, and a neighbourhood of 0 in $G \cap E_B$. If $\bar{U}_1^*$ is the closure of U_1 in E_B, we have $\bar{U} \cap E_B \supset \bar{U}_1^*$ because the topology of E_B is finer than that induced on the space by the topology on E.

Let P be the linear hull of $\bar{U}_1^*$. Then P is closed and of finite co-dimension in E_B. If $P = E_B$, $\bar{U} \cap E_B \supset \bar{U}_1^*$ and $\bar{U}_1^*$ is a neighbourhood of 0 in E_B. If $P \neq E_B$, and Q is the linear hull of a co-base of P in E_B, then let p and $q = I - p$ be the projections of E_B onto P and Q, where I is the identity on E_B. Then p and q are continuous. Since $\bar{U}$ generates L, $\bar{U} \cap E_B$ is absorbent in E_B. Let C be a basis of Q such that $C \subset \bar{U}$. It is easy to show that $\bar{U}_1^* + \Gamma C$ is bornivorous in E_B, and consequently $\bar{U} \cap E_B$ is bornivorous in E_B, so that $\bar{U}$ is bornivorous in L.

If $L \neq E$, then we still have that L is dense in E. The last lemma holds for the case when G has finite co-dimension in E. So there is a $B_o \subset E$ which is bounded closed and absolutely convex and $L \cap B_o$ is not closed in E. If $x \in \overline{(L \cap B_o)} \sim (L \cap B_o)$, then $x \in \overline{(L \cap B_o)} \subset \overline{B_o} = B_o$; so $x \notin L$. Because $\bar{U}$ is bornivorous in L, and $B_o \cap L$ is bounded there, there exists $\lambda > 0$ with $L \cap B_o \subset \lambda\bar{U} \Rightarrow \overline{L \cap B_o} \subset \lambda\bar{U} \subset L$. We have a contradiction, showing $L = E$.

From this, $\bigcap_{n=1}^{\infty} \overline{U_n} \supset \overline{(\bigcap_{n=1}^{\infty} U_n)} = \bar{U}$, and hence $\bigcap_{n=1}^{\infty} \overline{U_n}$ is bornivorous in E. Since E is a (DF)-space, $\bigcap_{n=1}^{\infty} \overline{U_n}$ is a neighbourhood of 0 in E, and $(\bigcap_{n=1}^{\infty} \overline{U_n}) \cap G = \bigcap_{n=1}^{\infty} U_n = U$ is a neighbourhood of 0 in G. So G is again a (DF)-space.

3) G *is arbitrary*. Let $\bar{G}$ be the closure of G in E.

If $\bar{G}$ is of finite co-dimension in E, we apply theorem 2.16 to conclude that $\bar{G}$ is a (DF)-space. If $\bar{G}$ is of infinite countable co-dimension, we apply stage (1) of the present theorem. Because $\bar{G}$ is closed in E, G is dense in $\bar{G}$, so we apply either theorem 2.16 or stage (2) to conclude G is a (DF)-space. #

2.21 THEOREM Let E be a barrelled (DF)-space. If G is a subspace of E of infinite countable co-dimension, then G is a (DF)-space.

PROOF VALDIVIA [20] proved that a subspace of infinite countable co-dimension of a barrelled space is barrelled. Since G is a subspace of the (DF)-space E, G has a fundamental sequence of bounded sets; consequently, G is a (DF)-space. #

We conclude this section with the following result.

2.22 THEOREM Let E be a sequentially complete (DF)-space. If G is a subspace of E of infinite countable co-dimension, then G is a (DF)-space.

PROOF Let $\mathcal{B}$ be the family of all bounded closed absolutely convex subsets of E. Let $E[\tau]$ be the locally convex hull of the normed spaces E_B where $B \in \mathcal{B}$, and the norm is the gauge of B. Because B is closed in E, it is sequentially complete as a subspace; by [13; 20,11.(2)] E_B is a Banach space. We have also that $E[\tau]$ is the bornological space associated with the initial space E.

Let E_1 be the closure of G in $E[\tau]$ with topology induced by the initial space E. We show that E_1 is a (DF)-space. If the co-dimension of E_1 in E is finite, the proof follows from theorem 2.16; so we assume E_1 has infinite countable co-dimension in E. Let $\{x_1, x_2, \ldots\}$ be a co-base of E_1 in E and E_n, for $n \geq 2$, be the linear hull of $E_1 \cup \{x_1, x_2, \ldots, x_{n-1}\}$. By [13; 27,1.(3)], $E[\tau]$ is barrelled since each E_B is barrelled. It is shown in [20] that if $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of subspaces of the barrelled space E such that $\bigcup_{n=1}^{\infty} E_n = E$, then E is the strict inductive limit of the sequence. If for each n, $\tau_n = \tau | E_n$, then $E[\tau]$ is the strict inductive limit of the spaces $\{E_n[\tau_n]\}$. Because E_1 is closed in $E[\tau]$ and E_1 is of finite co-dimension in E_n, by [9; Ch.2, Prop.10.3] we have E_n is closed in $E[\tau]$. Given any $B \in \mathcal{B}$, and noting [9; Ch.2, Th.12.2], there exists m with $B \subset E_m$. The conditions of theorem 2.20 are satisfied for the subspace E_1 of E; consequently, E_1 is a (DF)-space.

If G has finite co-dimension in E_1, then it is a (DF)-space; so we again assume G is of infinite countable co-dimension in E_1. Let $\{U_n\}_{n=1}^{\infty}$

be a sequence of closed absolutely convex neighbourhoods of 0 in G such that $U = \bigcap_{n=1}^{\infty} U_n$ is bornivorous. If τ' is $\tau|G$, we shall show first that U is a neighbourhood of 0 in $G[\tau']$.

U is a barrel in G, and hence also a barrel in $G[\tau']$. By the result of VALDIVIA mentioned earlier, $G[\tau']$ is barrelled, so that U is a neighbourhood of 0 in $G[\tau']$. Because E_1 is the closure of G in $E[\tau]$, G is dense in $E_1[\tau_1]$. Since the initial topology on E restricted to E_1 is coarser than τ_1, then $\bar{U}$ the closure of U in E_1 is a τ_1-neighbourhood of 0. Consequently $\bar{U}$ is bornivorous in E_1.

Let $\overline{U_n}$, for $n = 1,2, \ldots$, be the closure of U_n in E_1. Then $\bigcap_{n=1}^{\infty} \overline{U_n} \supset (\overline{\bigcap_{n=1}^{\infty} U_n}) = \bar{U}$, and so is bornivorous. Since E_1 is a (DF)-space, $\bigcap_{n=1}^{\infty} \overline{U_n}$ is a neighbourhood of 0 in E_1. We have

$(\bigcap_{n=1}^{\infty} \overline{U_n}) \cap G = \bigcap_{n=1}^{\infty} (\overline{U_n} \cap G) = \bigcap_{n=1}^{\infty} U_n = U$ is a neighbourhood of 0 in G. #

7. Bornological (DF)-spaces

These spaces are an important class of (DF)-spaces. In this section a topology is given on the dual of a metrizable space under which the space is a bornological (DF)-space. Conditions for a strong dual of a metrizable space to be bornological are given. The section concludes with an example of a quasi-barrelled (DF)-space which is not bornological.

The first result of the section identifies the bornological topology associated with the strong topology on the dual E' of a metrizable space E with the strong topology on E' derived from the pairing $\langle E', E'' \rangle$.

2.23 THEOREM Suppose that $E[\tau]$ is a locally convex metrizable space. The bornological topology τ_b^* associated with the strong topology $\tau_b(E)$

on E' is equal to $\tau_b(E'')$.

$E'[\tau_b(E'')]$ is thus always a complete bornological (DF)-space.

PROOF First, it is shown that $E'[\tau_b^*]$ has a basis of $\tau_b(E)$-closed neighbourhoods of 0. Let $\{B_n\}$ be a fundamental sequence of absolutely convex bounded $\tau_s(E)$-closed subsets of $E'[\tau_b(E)]$. We prove that the algebraic hull $V^a = \{y \in E' ;$ there exists $x \in V$ with $[x,y) \subset V\}$ of any set V of the form $V = \Gamma_{n=1}^{\infty} \alpha_n B_n$, with each $\alpha_n > 0$, is $\tau_b(E)$-closed.

Let $V_k = \Gamma_{n=1}^{k} \alpha_n B_n$, then $V = \bigcup_{k=1}^{\infty} V_k$. Because $E[\tau]$ is metrizable, the sets B_n are $\tau_s(E)$-compact and so are the V_k by [13; 20,6.(5)]. Hence the V_k are $\tau_b(E)$-closed. It is easily shown that $V^a \subset V^{oo}$ where polars are taken in the pairing $\langle E', E''\rangle$. By [13; 16,4.(4)] if $u \notin V^a$, there exists $\beta > 1$ with $u \notin \beta V$; consequently $\frac{u}{\beta} \notin V_k$ for each k. So for each k there exists $z_k \in V_k^o \subset E''$ with $z_k(u) = \beta$. The sequence $\{z_k\}$ is bounded in $E''[\tau_n(E')]$, and $\tau_n(E') = \tau_b(E')$; so, because $E'[\tau_b(E)]$ is a (DF)-space, the sequence $\{z_k\}$ is equicontinuous, so a relatively $\tau_s(E')$-compact subset of E''. So there is a $\tau_s(E')$-convergent subnet of $\{z_k\}$ which converges to some $z_o \in E''$. Because $z_k(u) = \beta$ for all k, $z_o(u) = \beta$. For each m we can choose a subnet of the $\tau_s(E')$-convergent net which takes all its values in the set $\{z_k\}_{k=m}^{\infty}$. Since $\{z_k\}_{k=m}^{\infty} \subset V_m^o$, we have $z_o \in \bigcap_{k=1}^{\infty} V_k^o = (\bigcup_{k=1}^{\infty} V_k)^o = V^o$. So $u \notin V^{oo}$; hence the $\tau_b(E)$-closure V^{oo} of V is equal to V^a.

Each V is a neighbourhood of 0 in τ_b^* because every bounded set is absorbed by sets of this form. If W is a τ_b^*-closed τ_b^*-neighbourhood of 0, then some V is contained in W. Obviously V^a is also contained in W, hence the sets of the form V^a form a basis of τ_b^*-neighbourhoods of

0 in E'.

The $\tau_b(E)$-closed absolutely convex absorbent subsets of E' form a basis of $\tau_b(E'')$-neighbourhoods of 0 in $E'[\tau_b(E'')]$. So we have τ_b^* is coarser than $\tau_b(E'')$. Conversely if A is a $\tau_b(E)$-closed absolutely convex absorbent subset of E', then A is a bornivorous barrel in $E'[\tau_b(E)]$ which is a complete space. Consequently A is a neighbourhood of 0 in $E'[\tau_b^*]$, so $\tau_b^* = \tau_b(E'')$.

Because $E'[\tau_b(E'')]$ has a basis of $\tau_b(E)$-closed neighbourhoods of 0, completeness of this space follows from [13; 18,4.(4)]. #

As a corollary to this we give some equivalent criteria for the strong dual of a metrizable space to be bornological.

2.24 COROLLARY Suppose that $E[\tau]$ is a locally convex metrizable space. The following are equivalent.

a) $E[\tau]$ is distinguished;

b) $E'[\tau_b(E)]$ is bornological; and

c) $E'[\tau_b(E)]$ is infrabarrelled.

PROOF Because $E'[\tau_b(E)]$ is complete, if it is quasi-barrelled, it is barrelled; so, (a) and (c) are equivalent. In the previous proof, it was shown that $E'[\tau_b^*]$ has a basis of neighbourhoods consisting of $\tau_b(E)$-closed absolutely convex sets. Because $E'[\tau_b(E)]$ is barrelled, (b) follows from (a). #

From this corollary, and results of section 3, the following is immediate.

2.25 COROLLARY Let $E[\tau]$ be an (F)-space.

a) If $E'[\tau_b(E)]$ is separable, then $E'[\tau_b(E)]$ is bornological.

b) If $E'[\tau_b(E)]$ has its bounded subsets metrizable, then $E'[\tau_b(E)]$ is bornological,

c) If $E[\tau]$ is reflexive, then $E'[\tau_b(E)]$ is bornological. #

GROTHENDIECK [7] asked whether each infrabarrelled (DF)-space is bornological. KŌMURA [11] gave an example of an infrabarrelled (DF)-space which was not bornological, but this example was incorrect. Recently VALDIVIA [22] has given an example of such a space. We state the following result which will give the desired example.

"Let λ be an echelon space defined by the increasing system $a^{(k)}$, $k = 1,2, \ldots,$ such that every $a^{(k)} = \{a_n^{(k)}\}_{n=1}^{\infty}$ is a sequence of strictly positive numbers. Suppose that for every positive integer p, there is a strictly increasing sequence $\{p_n\}_{n=1}^{\infty}$ of positive integers such that

1) $\{a_{p_n}^{(k)}/a_{p_n}^{(1)}\}_{n=1}^{\infty}$ is a bounded sequence for $k = 1,2, \ldots p$.

2) $\lim_{n\to\infty} (a_{p_n}^{(p+1)}/a_{p_n}^{(1)}) = \infty$.

Then there is a dense subspace E of $\lambda^*[\tau_s(\lambda)]$ such that $E[\tau_b(\lambda)|E]$ is a non-bornological infrabarrelled space."

The example is now obtained from the above result. We wish to construct the sequence $a^{(k)}$. Given positive n, we can write n uniquely as $(2m-1)2^{h-1}$ where m and h are positive integers. If $h \leq k$, set $a_n^{(k)} = n$; if $h > k$, set $a_n^{(k)} = 1$. Obviously the system of sequences is increasing. Given a positive integer p set $p_n = (2n-1)2^p$, $n = 1,2, \ldots$. From the definition, if $p+1 > k$, then $a_{p_n}^{(k)} = 1$. So, for $k = 1,2, \ldots p$, $a_{p_n}^{(k)} = 1$, and for $k = p+1$, $a_{p_n}^{(p+1)} = p_n$. The conditions of the theorem are satisfied, so we obtain a space $E[\tau_b(\lambda)|E]$ which is infrabarrelled and non-bornological. Since E with this topology is a subspace of the (DF)-space $\lambda^*[\tau_b(\lambda)]$, it has a fundamental sequence of bounded sets, and is hence a (DF)-space.

CHAPTER III

ULTRA-(DF)-SPACES

8 Definition and some properties of Ultra-(DF)-spaces

In the previous chapter we studied the class of (DF)-spaces motivated by the properties of strong duals of (F)-spaces. Since duality theory played an important role, local convexity was crucial in all the results. Based on a recent work of ERNST [6] we develop a class of topological vector spaces called Ultra-(DF)-spaces. The concept is meaningful in the setting of arbitrary topological vector spaces, which are not necessarily locally convex. Ultra-(DF)-spaces form a generalization of (DF)-spaces and most of the results available for (DF)-spaces carry over to this class with greater generality. The generalization from (DF)-space to Ultra-(DF)-space is patterned after the study of Ultra-barrelled and Ultra-quasibarrelled spaces by IYAHEN [10] in the context of a general topological vector space. In what follows $E[\tau]$ will refer to an arbitrary topological vector space (not necessarily locally convex) over the field of real or complex numbers.

We first recast the definition of a (DF)-space as follows to suit ready generalizations.

3.1 REMARK A locally convex space $E[\tau]$ is a (DF)-space provided

a') E has a fundamental sequence $\{B_n\}$ of closed absolutely convex bounded sets such that for each n, $B_n + B_n \subset B_{n+1}$ and

b') if $\{U_n\}$ is a sequence of closed absolutely convex τ-neighbourhoods of 0 in E and $U = \bigcap_{n=1}^{\infty} U_n$ absorbs all bounded sets in $E[\tau]$, then U is a τ-neighbourhood of 0. #

It is easily seen that the above definition is identical with the one we used in chapter II (cf. 2.1).

3.2 DEFINITION Let $M^{(0)}$ be a subset of a linear space E. The sequence $\{M^{(j)}\}_{j=1}^{\infty}$ is called a *defining sequence* for $M^{(0)}$ if $M^{(j)} \subset E$ and $M^{(j)} + M^{(j)} \subset M^{(j-1)}$ for each $j = 1,2, \ldots$. #

We now give the definition of an Ultra-(DF)-space.

3.3 DEFINITION A Hausdorff topological vector space $E[\tau]$ is called an *Ultra-(DF)-space* (or (UDF)-space) if

a) there exists a fundamental sequence $\{B_n\}$ of circled closed bounded sets in $E[\tau]$ and

b) if $\{U_n^{(0)}\}_{n=1}^{\infty}$ is a sequence of closed circled neighbourhoods of 0 in $E[\tau]$ and $\{U_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $U_n^{(0)}$, for each n, and for each $j \geq 0$, $U^{(j)} = \bigcap_{n=1}^{\infty} U_n^{(j)}$ absorbs all bounded sets in the fundamental sequence, then $U^{(0)}$ is a τ-neighbourhood of 0 in E. #

Given a topological vector space $E[\tau]$ there is a finest locally convex topology τ^{oo} on E which is coarser than τ. A neighbourhood basis of 0 for τ^{oo} consists of convex hulls of members of a τ-neighbourhood basis of 0. In case τ^{oo} is Hausdorff, following IYAHEN [10], we call $E[\tau^{oo}]$ the *locally convex space associated* with $E[\tau]$.

As a first result, we show that every locally convex (UDF)-space is a (DF)-space.

3.4 THEOREM If $E[\tau]$ is a (UDF)-space with a fundamental sequence $\{B_n\}$ of closed circled bounded sets and the topology τ^{oo} is Hausdorff, then

a) the sequence $\{C_n\}$ where $C_n = \overline{\langle B_n \rangle}$ (in τ^{oo}) is a fundamental sequence of absolutely convex closed bounded sets in $E[\tau^{oo}]$ and

b) $E[\tau^{oo}]$ is a (DF)-space.—

PROOF We first prove 3.3(b). Let $\{W_n\}$ be a sequence of absolutely convex closed τ^{oo}-neighbourhoods of 0 and let $W = \bigcap_1^\infty W_n$. It suffices to prove that W is a τ^{oo}-neighbourhood of 0 provided it absorbs each B_n.

Each W_n is a circled closed τ-neighbourhood of 0 and has a defining sequence $\{\frac{1}{2^j} W_n\}_{j=1}^\infty$. If W absorbs all B_n, so does $\frac{1}{2^j} W = \bigcap_{n=1}^\infty \frac{1}{2^j} W_n$ and since it is convex, W is a τ^{oo}-neighbourhood of 0.

Coming to the proof of (a), each C_n is obviously bounded in $E[\tau^{oo}]$. We first show that each bounded set C in $E[\tau^{oo}]$ is absorbed by some C_n. If not, there exists $x_n \in \frac{1}{n} C$, $x_n \notin C_n$ for each n. Clearly, $\{x_n\}$ is a τ^{oo}-null sequence. Since each C_n is closed and each x_n is compact, by [13; 15,6.(9)] there is an absolutely convex τ^{oo}-neighbourhood V_n of 0 such that $x_n \notin \overline{C_n + V_n}^{\tau^{oo}}$, for each n. Now $W_n = \overline{C_n + V_n}^{\tau^{oo}}$ is a closed absolutely convex τ^{oo}-neighbourhood of 0 and $W = \bigcap_1^\infty W_n$ does not contain any of the x_n's.

For a fixed m, $B_m \subset C_{m+n} \subset W_{m+n}$ for $n = 0,1,2, \ldots,$ so $B_m \subset \bigcap_{n=m}^\infty W_n$. But $\bigcap_{n=1}^{m-1} W_n$ is a τ^{oo}-neighbourhood of 0, hence $W = (\bigcap_{n=1}^{m-1} W_n) \cap (\bigcap_{n=m}^\infty W_n)$ absorbs B_m. Hence W is a τ^{oo}-neighbourhood of 0 which is a contradiction.

We now show that C lies in some C_n. Clearly $C \subset \lambda C_m$ for some $\lambda > 0$. Since $\{B_n\}$ is a fundamental sequence of bounded sets in $E[\tau]$, $\lambda B_m \subset B_n$ for some n; hence, $C \subset \lambda C_m \subset C_n$. This completes the proof. #

As an immediate consequence we have

3.5 COROLLARY If $E[\tau]$ is a locally convex (UDF)-space then it is a (DF)-space. #

The proof is now obvious.

Let us now recall the definition of an Ultra-barrel and an Ultra-quasi-barrel from IYAHEN [10].

3.6 DEFINITION A circled closed set $U^{(0)}$ of a Hausdorff topological vector space $E[\tau]$ with a defining sequence $\{U^{(j)}\}_{j=1}^{\infty}$ such that each $U^{(j)}$ is absorbent is called an *Ultra-barrel*. If each Ultra-barrel is a τ-neighbourhood of 0, then $E[\tau]$ is said to be an *Ultra-barrelled* space.

If in the above statement, each $U^{(j)}$ absorbs every bounded set then $U^{(0)}$ is called an *Ultra-quasibarrel*, and if each Ultra-quasibarrel is a τ-neighbourhood of 0 then $E[\tau]$ is said to be *Ultra-quasibarrelled*. #

In 3.3(b) the sequence $\{U^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $U^{(0)}$ and so every Ultra-quasibarrelled space with a fundamental sequence of bounded sets is a (UDF)-space. Since a *locally bounded space* is defined as a Hausdorff topological vector space with a bounded neighbourhood of 0 it is Ultra-quasibarrelled and hence a (UDF)-space. We shall now see that for separable (UDF)-spaces, the converse is also true.

3.7 THEOREM Let $E[\tau]$ be a (UDF)-space and $M \subset E$ be a separable set containing 0. If $V^{(0)}$ is an Ultra-quasibarrel with defining sequence $\{V^{(j)}\}_{j=1}^{\infty}$ as in 3.6, then $V^{(0)} \cap M$ is a neighbourhood of 0 in the topology induced by τ on M.

PROOF It suffices to show that there exists an open neighbourhood U of 0 in $E[\tau]$ such that $M \cap U \subset V^{(0)}$, that is $U \cap M \cap (E \sim V^{(0)}) = \phi$. Again since $U \cap (E \sim V^{(0)})$ is open, it is enough to show that no element of a dense sequence $\{x_n\}$ in M lies in $U \cap (E \sim V^{(0)})$. If $x_n \in E \sim V^{(0)}$ we shall show that $x_n \notin U$. If only finitely many x_n belong to $E \sim V^{(0)}$ the existence of U is obvious. If not, we let $\{x_n\}_{n=1}^{\infty}$ represent the subsequence of elements in $E \sim V^{(0)}$.

Since $V^{(0)}$ is closed, for each n there is a circled neighbourhood $\tilde{U}_n^{(0)}$ of 0 with a defining sequence $\{\tilde{U}_n^{(j)}\}_{j=1}^{\infty}$ of neighbourhoods of 0 so that $x_n \notin \overline{V^{(0)} + \tilde{U}_n^{(0)}}$. The set $U_n^{(j)} = \overline{V^{(j)} + \tilde{U}_n^{(j)}}$ is a closed neighbourhood of 0 for each j and is circled for $j = 0$. Again $\{U_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $U_n^{(0)}$ for each n. Because $U^{(j)} = \bigcap_{n=1}^{\infty} U_n^{(j)}$ contains $V^{(j)}$ for each j, $U^{(j)}$ absorbs all bounded sets. Hence $U^{(0)}$ is a closed neighbourhood of 0 which contains no x_n. Hence the existence of an open U which we are seeking is guaranteed. #

As an immediate corollary we have

3.8 COROLLARY

a) Every Ultra-quasibarrelled space with a fundamental sequence of bounded sets is a (UDF)-space.

b) Every separable (UDF)-space is Ultra-quasibarrelled. #

9. <u>Linear maps on (UDF)-spaces</u>.

Just as in the case of (DF)-spaces it will now be shown that a linear map from a (UDF)-space into a topological vector space is continuous provided it is continuous on each bounded set. In this section, whenever we refer to a fundamental sequence $\{B_n\}$ we shall make the blanket assumption that $B_n + B_n \subset B_{n+1}$ for each n.

3.9 LEMMA Let $E[\tau]$ be a (UDF)-space with a fundamental sequence $\{B_n\}$ of circled closed bounded sets and let $V^{(0)}$ be a circled subset of E with a defining sequence $\{V^{(j)}\}_{j=1}^{\infty}$. Then $V^{(0)}$ is a τ-neighbourhood of 0 if $V^{(j)} \cap B_n$ is a neighbourhood of 0 in the induced topology on B_n for each $n \geq 1$ and $j \geq 0$.

PROOF For each n, there is a circled closed neighbourhood $U_n^{(0)}$ of 0, such that $B_{n+1} \cap U_n^{(0)} \subset B_{n+1} \cap V^{(1)}$. Obviously, $B_n \cap U_n^{(0)} \subset B_n \cap V^{(1)}$. It is easily seen that for each $U_n^{(0)}$ there is a defining sequence $\{U_n^{(j)}\}$ of circled closed neighbourhoods of 0 such that $B_n \cap U_n^{(j)} \subset B_n \cap V^{(j+1)}$ for each $j \geq 1$. Since $B_{n-j} \subset B_n$ for $j < n$, we have $B_{n-j} \cap U_n^{(j)} \subset B_{n-j} \cap V^{(j+1)}$ for $n \geq 1$ and $0 \leq j < n$.

We shall now construct a sequence $\{W_n^{(0)}\}_{n=1}^{\infty}$ of circled closed neighbourhoods of 0 such that each $W_n^{(0)}$ has a defining sequence $\{W_n^{(j)}\}_{j=1}^{\infty}$ with the property that the sets $W^{(j)} = \bigcap_{n=1}^{\infty} W_n^{(j)}$ $(j \geq 0)$ absorb all bounded sets.

$$\text{Let} \quad W_n^{(j)} = \begin{cases} \overline{B_{n-j} \cap V^{(j+1)} + U_n^{(j+1)}} & \text{for } j = 0, \ldots, n-1 \\ U_n^{(j+1)} & \text{for } j = n, n+1, \ldots \end{cases}$$

We may assume that $V^{(j)}$ is circled for each j; clearly each $W_n^{(j)}$ is a circled closed neighbourhood of 0.

For $1 \leq j < n$

$$\begin{aligned} W_n^{(j)} + W_n^{(j)} &= \overline{B_{n-j} \cap V^{(j+1)} + U_n^{(j+1)}} + \overline{B_{n-j} \cap V^{(j+1)} + U_n^{(j+1)}} \\ &\subset \overline{B_{n-j} \cap V^{(j+1)} + B_{n-j} \cap V^{(j+1)} + U_n^{(j+1)} + U_n^{(j+1)}} \\ &\subset \overline{B_{n-(j-1)} \cap V^{(j)} + U_n^{(j)}} = W_n^{(j-1)}. \end{aligned}$$

If $j = n$, $W_n^{(n)} + W_n^{(n)} = U_n^{(n+1)} + U_n^{(n+1)} \subset U_n^{(n)} \subset W_n^{(n-1)}$,

and if $j > n$, $W_n^{(j)} + W_n^{(j)} \subset W_n^{(j-1)}$ as $\{U_n^{(j)}\}_{j=1}^{\infty}$ forms a defining sequence for $U_n^{(0)}$. Thus $\{W_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $W_n^{(0)}$.

Set $W^{(j)} = \bigcap_{n=1}^{\infty} W_n^{(j)}$. For arbitrary but fixed j, there is a number $\rho_n > 0$ with $B_{n-j} \subset \rho_n(B_{n-j} \cap U_n^{(j)})$ for $n > j$. For $p = 0,1,2,\ldots$,

$$B_{n-j} \subset \rho_n(B_{n-j} \cap U_n^{(j)}) \subset \rho_n(B_{n-j} \cap V^{(j+1)})$$

$$\subset \rho_n(B_{n-j+p} \cap V^{(j+1)}) \subset \rho_n W_{n+p}^{(j)}.$$

Hence $B_{n-j} \subset \rho_n \bigcap_{p=0}^{\infty} W_{n+p}^{(j)}$. But $W^{(j)} = (\bigcap_{m=1}^{n-1} W_m^{(j)}) \cap (\bigcap_{p=0}^{\infty} W_{n+p}^{(j)})$ and $\bigcap_{m=1}^{n-1} W_m^{(j)}$ absorbs B_{n-j}. So it follows that $W^{(j)}$ absorbs all B_{n-j} for $n > j$ and hence absorbs all B_n.

Finally we show that $W^{(0)} \subset V^{(0)}$ to conclude that $V^{(0)}$ is a neighbourhood of 0 since $E[\tau]$ is a (UDF)-space. Now

$$W_n^{(0)} = B_n \cap V^{(1)} + U_n^{(1)} \subset B_n \cap V^{(1)} + U_n^{(1)} + U_n^{(1)}$$

$$\subset B_n \cap V^{(1)} + U_n^{(0)}, \text{ so}$$

$$B_n \cap W_n^{(0)} \subset B_n \cap (B_n \cap V^{(1)} + U_n^{(0)})$$

$$\subset \{B_n \cap V^{(1)} + [(B_n + B_n) \cap U_n^{(0)}]\}$$

$$\subset (B_{n+1} \cap V^{(1)}) + (B_{n+1} \cap U_n^{(0)}) \subset (B_{n+1} \cap V^{(1)}) + (B_{n+1} \cap V^{(1)}).$$

So it follows for $n = 1,2,\ldots$,

$$B_n \cap W^{(0)} \subset (B_{n+1} \cap V^{(1)}) + (B_{n+1} \cap V^{(1)}).$$

Because $\bigcup_{n=1}^{\infty} B_{n+k} = E$ for any $k \geq 0$,

$$W^{(0)} = \bigcup_{n=1}^{\infty} B_n \cap W^{(0)} \subset \bigcup_{n=1}^{\infty} \{(B_{n+1} \cap V^{(1)}) + (B_{n+1} \cap V^{(1)})\}$$

$$\subset \bigcup_{n=1}^{\infty} B_{n+1} \cap V^{(1)} + \bigcup_{n=1}^{\infty} B_{n+1} \cap V^{(1)}$$

$$= V^{(1)} + V^{(1)} \subset V^{(0)}$$ completing the proof. #

Using 3.9 we can now prove the following result.

3.10 THEOREM A family A of linear maps from a (UDF)-space $E[\tau]$ with a fundamental sequence $\{B_n\}_{n=1}^{\infty}$ of circled, closed bounded sets into an arbitrary topological vector space $F[\tau']$ is equicontinuous if and only if for each n the restriction of A to B_n is equicontinuous at 0.

Consequently a linear map from $E[\tau]$ into $F[\tau']$ is continuous if and only if it is continuous at 0 in each B_n.

PROOF Let $U^{(0)}$ be a circled neighbourhood of 0 in $F[\tau']$ and $V^{(0)} = \bigcap_{f \in A} f^{-1}[U^{(0)}]$. Obviously $V^{(0)}$ is circled. Let $\{U^{(j)}\}_{j=1}^{\infty}$ be a defining sequence of $U^{(0)}$, neighbourhoods of 0 in $F[\tau']$. Define $V^{(j)} = \bigcap_{f \in A} f^{-1}[U^{(j)}]$. Then $\{V^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $V^{(0)}$. Now $V^{(j)} \cap B_n = \bigcap_{f \in A} f^{-1}(U^{(j)}) \cap B_n$ is a neighbourhood of 0 in B_n and so by 3.9 $V^{(0)}$ is a neighbourhood of 0 in $E[\tau]$. This shows that A is equicontinuous. #

The concepts of $(*)$-inductive limit and strict $(*)$-inductive limit for an arbitrary topological vector space were introduced by IYAHEN [10]. Using them we shall now represent every (UDF)-space as a strict $(*)$-inductive limit of a suitable family of topological vector spaces.

3.11 DEFINITION Suppose that E is a linear space, $\{E_\alpha\}_{\alpha \in A}$ is a class of topological vector spaces, and $\{u_\alpha\}_{\alpha \in A}$ is a class of linear maps where

$u_\alpha : E_\alpha \to E$, and the span of $\bigcup_{\alpha\in A} u_\alpha(E_\alpha)$ is E. Then E with the finest vector topology such that all u_α are continuous is called the *(*)-inductive limit of the spaces* $\{E_\alpha\}_{\alpha\in A}$ *by the mappings* $\{u_\alpha\}_{\alpha\in A}$. #

3.12 DEFINITION Suppose that E is a linear space, and $\{E_n\}_{n\in\mathbb{N}}$ is a class of subspaces of E such that $E_n \subset E_{n+1}$, for each n, $\{I_n\}_{n\in\mathbb{N}}$ is the set of injections $I_n : E_n \to E$, and $\bigcup_{n\in\mathbb{N}} E_n = E$. If, for each n, the topology on E_n coincides with that induced by E_{n+1}, then the (*)-inductive limit of the spaces $\{E_n\}_{n\in\mathbb{N}}$ by the mappings $\{I_n\}_{n\in\mathbb{N}}$ is called the *strict (*)-inductive limit* of the spaces $\{E_n\}_{n\in\mathbb{N}}$. #

3.13 THEOREM Let $E[\tau]$ be a (UDF)-space with a fundamental sequence $\{B_n\}_{n=1}^{\infty}$ of bounded sets. For $n = 1,2, \ldots$ let E_n be the linear span of B_n with the induced topology τ_n from E, then $E[\tau]$ is the strict (*)-inductive limit of the spaces $\{E_n[\tau_n]\}_{n\in\mathbb{N}}$.

PROOF The (*)-inductive limit τ' of these spaces, by definition, is finer than τ and the canonical maps

$I_n : E_n[\tau_n] \to E[\tau]$ are continuous. Since $I \circ I_n$ is the restriction of the identity mapping $I : E[\tau] \to E[\tau']$ to E_n, it follows I is continuous on a fundamental sequence of bounded sets. By 3.10, I is continuous and hence $\tau = \tau'$. Since $E = \bigcup_1^\infty E_n$ the desired result follows. #

As in the case of (DF)-spaces one can prove the following.

3.14 THEOREM If $E[\tau]$ is a (UDF)-space with a fundamental sequence $\{B_n\}$ of bounded sets such that the restriction of τ to each B_n is metrizable, then E is Ultra-quasibarrelled.

PROOF We can assume that each B_n is circled and closed. Let $V^{(0)}$ be an Ultra-quasibarrel, with a defining sequence $\{V^{(j)}\}_{j=1}^{\infty}$ of sets such that $V^{(j)}$ absorbs each B_n. We may assume that the $V^{(j)}$ are closed and circled. Each $V^{(j)}$ is an Ultra-quasibarrel because $\{V^{(n+j)}\}_{n=1}^{\infty}$ is a defining sequence. Since a finite intersection of Ultra-quasibarrels is again an Ultra-quasibarrel, the family of all Ultra-quasibarrels forms a basis of neighbourhoods of 0 for a vector topology τ' on E finer than τ.

Using 3.7, we may show that τ and τ' coincide on separable subsets of $E[\tau]$. As a consequence of the proof of theorem 2.7(b), we have that τ and τ' coincide on the bounded subsets of $E[\tau]$. Noting that each Ultra-quasibarrel has a defining sequence of Ultra-quasibarrels, we conclude from lemma 3.9, that each Ultra-quasibarrel is a τ-neighbourhood of 0 in E. #

As we have already seen, 3.10 gives an important property of (UDF)-spaces. One would be curious to know if this property of linear maps could characterize (UDF)-spaces or (DF)-spaces. The following counter-example settles this question in the negative ([6]).

Let $E[\tau]$ be locally convex metrizable, so that $E'[\tau_b(E)]$ is a (DF)-space with a fundamental sequence $\{B_n\}$ of absolutely convex closed bounded sets. We can further assume that for each n, $B_n + B_n \subset B_{n+1}$ and B_n is $\tau_s(E)$-closed. J.B. COOPER [3] has considered the topology τ' on E' defined as follows.

Let $\{U_n^*\}_{n=1}^{\infty}$ be a sequence of absolutely convex $\tau_s(E)$-neighbourhoods of 0 and let $\eta(U_n^*) = \bigcup_{n=1}^{\infty} (U_1^* \cap B_1 + \dots + U_n^* \cap B_n)$. Then the set of all such $\eta(U_n^*)$ forms a basis of neighbourhoods of 0 for a locally convex topology τ' on E' which has the following properties.

3.15 PROPOSITION (COOPER [3])

a) $\tau_s(E) \leq \tau' \leq \tau_b(E)$.

b) On $\tau_b(E)$-bounded sets, τ' and $\tau_s(E)$ coincide.

c) A family A of linear maps from $E'[\tau']$ into another topological vector space F is equicontinuous if and only if its restriction to each B_n is equicontinuous.

d) τ' is the finest topology which agrees with $\tau_s(E)$ on τ-equicontinuous subsets of E'. #

Since $E[\tau]$ is locally convex metrizable, by the Banach-Dieudonné Theorem [13; 21,10.(1)], we have $\tau' = \tau_c(E)$ the topology of uniform convergence on the τ-precompact subsets of E.

In general $E'[\tau_c(E)]$ need not be a (DF)-space. For example let $E = \ell^2$. Then $(E'[\tau_c(E)])' = E$. We shall show that in ℓ^2 there exists a sequence $\{M_n\}$ of precompact sets whose union M is bounded but not precompact.

Let e_n be the nth co-ordinate vector. Then the sets $M_n = \{e_n\}$ are precompact and the set $M = \bigcup_{n=1}^{\infty} M_n$ is bounded. It is easy to show that M is not precompact in ℓ^2. Because $E'[\tau_c(E)]$ is not a (DF)-space, it is not a (UDF)-space.

Next we shall consider linear maps between (UDF)-spaces and metrizable spaces.

3.16 LEMMA If $E[\tau]$ is a (UDF)-space, for each sequence $\{U_n\}$ of τ-neighbourhoods of 0, there is a τ-neighbourhood U of 0 which is absorbed by all U_n.

PROOF Assume without loss of generality that each $U_n = \tilde{U}_n^{(0)}$ is circled and closed. Then each $\tilde{U}_n^{(0)}$ has a defining sequence $\{\tilde{U}_n^{(j)}\}_{j=1}^{\infty}$ of closed neighbourhoods of 0. It suffices to prove the assertion for $\{U_n^{(0)}\}_{n=1}^{\infty}$ where $U_n^{(0)} = \bigcap_{i=1}^{n} \tilde{U}_i^{(0)}$. Each $U_n^{(0)}$ is again a circled closed neighbourhood of 0 with defining sequence $\{U_n^{(j)}\}_{j=1}^{\infty}$ where $U_n^{(j)} = \bigcap_{i=1}^{n} \tilde{U}_i^{(j)}$. Let $\{B_n\}$ be a fundamental sequence of circled closed bounded sets. We can assume that B_1 is not contained in all multiples of $U_1^{(0)}$. [Otherwise, if some $U_n^{(0)}$ does not equal E, we can find an integer m so that $B_m \not\subset U_m^{(0)}$. It suffices to prove the theorem for the sequences $\{U_{m+k}^{(0)}\}_{k=1}^{\infty}$ where we take $\{B_{m+k}\}_{k=1}^{\infty}$ as a fundamental sequence of bounded sets.]

We will show that there exist positive numbers ρ_n such that $U^{(0)} = \bigcap_{n=1}^{\infty} \rho_n U_n^{(0)}$ is a neighbourhood. Again it is enough to show the existence of ρ_n such that for each $j \geq 0$ $\bigcap_{n=1}^{\infty} \rho_n U_n^{(j)} = U^{(j)}$ absorbs all B_n. Let ${}_nM_i^{(j)} = \sup\{\mu > 0 : \mu B_n \subset U_i^{(j)}\}$. By the assumption on B_1 and $U_1^{(0)}$, ${}_nM_i^{(j)} < \infty$. Let $\rho_i = \frac{1}{{}_iM_i^{(i)}}$ for $i = 1,2,\ldots$.

Since $U_i^{(j)} \supset U_i^{(j+1)}$, and $B_n \subset B_{n+1}$, we have ${}_nM_i^{(j)} \geq {}_nM_i^{(j+1)}$ and ${}_nM_i^{(j)} \geq {}_{n+1}M_i^{(j)}$, for each i,j,n. For fixed n and j, and arbitrary $i \geq n,j$; ${}_nM_i^{(j)} \geq {}_iM_i^{(i)}$. Hence, $\rho_i \, {}_nM_i^{(j)} \geq 1$ and so

${}_nK^{(j)} = \inf_{i \in \mathbb{N}} \{\rho_i \, {}_nM_i^{(j)}\} > 0$. Since $U_i^{(j)}$ is closed, we have ${}_nM_i^{(j)} B_n \subset U_i^{(j)}$, and so ${}_nK^{(j)} B_n \subset \rho_i U_i^{(j)}$.

Hence ${}_nK^{(j)} B_n \subset U^{(j)} = \bigcap_{i=1}^{\infty} \rho_i U_i^{(j)}$. So each $U^{(j)}$ absorbs all B_n. Consequently, $U^{(0)}$ is a neighbourhood of 0 which is absorbed by all $U_n^{(0)}$, and hence, by all U_n. #

3.17 DEFINITION. A family A of linear maps from a topological vector space $E[\tau]$ into a topological vector space $F[\tau']$ is *equibounded* if there exists a τ-neighbourhood U of 0 in E such that the set $A[U] = \bigcup_{f \in A} f[U]$ is bounded in the range space $F[\tau']$. #

3.18 THEOREM For any arbitrary topological vector space $E[\tau]$ the following statements are equivalent.

a) For each sequence $\{U_n\}_{n=1}^{\infty}$ of neighbourhoods in E, there is a neighbourhood U of 0 which is absorbed by each U_n.

b) Each equicontinuous set A of linear maps from $E[\tau]$ into each metrizable topological vector space $F[\tau']$ is equibounded.

Obviously, then, if $E[\tau]$ is a (UDF)-space, then (b) is true.

PROOF

Necessity: Let $\{V_n\}_{n=1}^{\infty}$ be a neighbourhood basis of 0 in $F[\tau']$. Then because the family A is equicontinuous, the set $A^{-1}[V_n] = \bigcap_{f \in A} f^{-1}[V_n] = U_n$ is a neighbourhood of 0 in $E[\tau]$ for each n. So, there is a neighbourhood U of 0 in $E[\tau]$ which is absorbed by all U_n. Consequently, $A[U]$ is absorbed by all V_n, i.e. A is equibounded.

Sufficiency: As in the previous lemma it suffices to show the result for a sequence of circled, closed neighbourhoods $\{\tilde{U}_n^{(0)}\}_{n=1}^{\infty}$. Each $\tilde{U}_n^{(0)}$ has a defining sequence $\{\tilde{U}_n^{(j)}\}_{j=1}^{\infty}$ of circled closed neighbourhoods of 0 in $E[\tau]$.

Define $U_1 = \tilde{U}_1^{(0)}$, and $U_n = \bigcap_{i=1}^{n} \tilde{U}_i^{(n)}$, for $n = 2,3, \ldots$. Then each U_n is a circled closed neighbourhood of 0 in $E[\tau]$, and it is easily seen that $U_{n+1} + U_{n+1} \subset U_n$ for $n = 1,2, \ldots$.

The sequence $\{U_n\}_{n=1}^{\infty}$ thus forms a neighbourhood basis at 0 for a pseudo-metrizable vector topology τ' on E. If $N = \bigcap_1^{\infty} U_n$, then N is a closed linear subspace of E, so that the quotient space $E/N[\tau'_q]$ is a metrizable space. The canonical mapping $K : E[\tau] \to E/N[\tau'_q]$ is a continuous linear map. Because K is equibounded, there is a τ-neighbourhood U of 0 in E such that $K[U]$ is bounded in $E/N[\tau'_q]$. So for each n, there is a $\rho_n > 0$ such that $K[U] \subset \rho_n [U_{n+1}]$. From this it follows that $U \subset \rho_n(U_{n+1} + N) \subset \rho_n U_n$. #

Since the identity map is always continuous, we have the following.

3.19 COROLLARY Every metrizable (UDF)-space is locally bounded. #

Now, we consider linear maps from pseudo-metrizable spaces into (UDF)-spaces.

3.20 THEOREM A set A of linear maps from a pseudo-metrizable space $F[\tau']$ into a (UDF)-space $E[\tau]$ is equicontinuous if and only if it is equibounded.

PROOF Let $\{V_n\}_{n=1}^{\infty}$ be a decreasing neighbourhood basis of 0 for $F[\tau']$ and $\{B_n\}_{n=1}^{\infty}$ be a fundamental sequence of bounded sets in $E[\tau]$. If A is not equibounded, $A[V_n] \not\subset B_n$, for $n = 1,2, \ldots$. So there exists $x_n \in V_n$ such that $A(x_n) = \bigcup_{f \in A} \{f(x_n)\} \not\subset B_n$ for $n = 1,2, \ldots$. The set $B = \bigcup_{n=1}^{\infty} A(x_n)$ is thus not bounded in $E[\tau]$. But $\{x_n\}_{n=1}^{\infty}$ is a null sequence in $F[\tau']$, hence bounded. Because A is equicontinuous, B is bounded, so we have a contradiction; this proves the necessity.

The sufficiency is obvious. #

10 The space $L_b(E,F)$

Let $L_b(E,F)$ denote the space $L(E,F)$ of continuous linear maps from a topological vector space E into a topological vector space F, with the topology of uniform convergence on all bounded sets of E. The aim of the section is to characterize (UDF)-spaces with the help of $L_b(E,F)$ where F is a metrizable space. This is parallel to the definition of (DF)-spaces via the dual space. We need a few notations. If $M \subset E$ and $N \subset F$, then we let $[M,N] = \{f : E \to F : f \text{ is a linear map and } f[M] \subset N\}$, and $\langle M,N\rangle = \{f \in L(E,F) : f[M] \subset N\}$.

We begin with a completeness criterion for $L_b(E,F)$.

3.21 THEOREM If $E[\tau]$ is a (UDF)-space and $F[\tau']$ is a complete Hausdorff topological vector space, then $L_b(E,F)$ is complete.

PROOF Let $\mathcal{B}$ be the class of all bounded sets in $E[\tau]$ and $\mathcal{V}$ be a neighbourhood basis of 0 for $F[\tau']$. N. ADASCH [1] has shown that the completion $\widetilde{L_b(E,F)}$ of $L_b(E,F)$ has the representation

$$\widetilde{L_b(E,F)} = \bigcap_{\substack{B\in\mathcal{B}\\ V\in\mathcal{V}}} (L(E,F) + [B,V]).$$ It suffices to show that if $A \in \widetilde{L_b(E,F)}$, then $A \in L(E,F)$; i.e. A is continuous.

Let $V,W \in \mathcal{V}$ such that $V + V \subset W$. Let B be circled and bounded in $E[\tau]$. Then there exists $A_1 \in L(E,F)$, $A_2 \in [B,V]$, and a neighbourhood U of 0 in $E[\tau]$ such that

$$A = A_1 + A_2 \text{ and } A_1[B \cap U] \subset V,\ A_2[B \cap U] \subset V.$$

Hence $A[B \cap U] \subset A_1[B \cap U] + A_2[B \cap U] \subset V + V \subset W$. So, A is continuous at 0 in B. By 3.10, $A \in L(E,F)$. #

Analogous to the definition of a (DF)space via dual spaces we have the following parallel result for (UDF)-spaces.

3.22 THEOREM Let $E[\tau]$ be a (UDF)-space, and $F[\tau']$ an arbitrary topological vector space. If $A \subset L_b(E,F)$ is bounded, and $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \subset L(E,F)$ is equicontinuous, then A is equicontinuous.

PROOF Let $V^{(0)}$ be a circled closed neighbourhood of 0 in F, and $\{V^{(j)}\}_{j=1}^{\infty}$ be a defining sequence of $V^{(0)}$, where each $V^{(j)}$ is a circled closed neighbourhood of 0 in $F[\tau']$.

Because each A_n is equicontinuous and each $V^{(j)}$ is a circled closed neighbourhood of 0 in $F[\tau']$,

$U_n^{(j)} = A_n^{-1}[V^{(j)}] = \bigcap_{f \in A_n} f^{-1}[V^{(j)}]$ is a circled closed neighbourhood of 0 in $E[\tau]$ for each n and j. Then

$$U_n^{(j+1)} + U_n^{(j+1)} = A_n^{-1}[V^{(j+1)}] + A_n^{-1}[V^{(j+1)}]$$

$$\subset A_n^{-1}[V^{(j)}] = U_n^{(j)} \quad \text{for } n \geq 1,$$

$j \geq 0$; so, for each n, the sequence $\{U_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $U_n^{(0)}$.

If suffices to show $U^{(j)} = \bigcap_{n=1}^{\infty} U_n^{(j)}$ absorbs all bounded sets for each $j \geq 0$ because $U^{(0)} = \bigcap_{f \in A} f^{-1}[V^{(0)}]$.

Let B be bounded in $E[\tau]$. Since A is bounded in $L_b(E,F)$, there exists a real number $\lambda_B^{(j)} > 0$, for each $j \geq 0$, such that

$A \subset \lambda_B^{(j)} \langle B, V^{(j)} \rangle$. Then we have

$\frac{1}{\lambda_B^{(j)}} A \subset \langle B, V^{(j)} \rangle$; so, $\frac{1}{\lambda_B^{(j)}} A[B] \subset V^{(j)}$, from which it follows that $A[\frac{1}{\lambda_B^{(j)}}B] \subset V^{(j)}$. Thus

$$\frac{1}{\lambda_B^{(j)}}B \subset A^{-1}[V^{(j)}] = U^{(j)}. \quad \#$$

As an immediate corollary we have

3.23 COROLLARY Let $E[\tau]$ be a (UDF)-space, and $F[\tau']$ an arbitrary topological vector space. Every bounded sequence and every bounded set, separable in the topology of pointwise convergence on $L(E,F)$, in $L_b(E,F)$ is equicontinuous. #

Since every (UDF)-space has a fundamental sequence of bounded sets, the next result is obvious.

3.24 LEMMA If $E[\tau]$ is a (UDF)-space and $F[\tau']$ is a pseudo-metrizable topological vector space, then $L_b(E,F)$ is a pseudo-metrizable topological vector space. #

We shall now prove partial converses of results 3.22 and 3.24.

3.25 LEMMA If $E[\tau]$ is a topological vector space and for each metrizable space $F[\tau']$ it is true that each bounded set $A \subset L_b(E,F)$ is equicontinuous which is a union of a sequence of equicontinuous sets $A_n \subset L(E,F)$, then $E[\tau]$ has property (b) of 3.3.

PROOF Let $\{U_n^{(0)}\}_{n=1}^{\infty}$ be a sequence of circled closed neighbourhoods of 0 in E such that each $U_n^{(0)}$ has a defining sequence $\{U_n^{(j)}\}_{j=1}^{\infty}$ of sets with each set $U^{(j)} = \bigcap_{n=1}^{\infty} U_n^{(j)}$ absorbing all bounded sets of E. We may assume each $U_n^{(j)}$ is circled. It must be shown that $U^{(0)} = \bigcap_{n=1}^{\infty} U_n^{(0)}$ is a neighbourhood of 0 in E.

We now construct a metrizable space $\hat{F}[\tau']$. Let $F = \bigoplus_{n=1}^{\infty} E_n$ be the algebraic direct sum of the spaces $\{E_n\}_{n=1}^{\infty}$ where each $E_n = E$. For $j \geq 0$ set $V^{(j)} = \bigoplus_{n=1}^{\infty} U_n^{(j)}$; then the sets $\{V^{(j)}\}$ form a neighbourhood basis for a pseudo-metrizable vector topology τ^* on F. Letting $N = \bigcap_{j=0}^{\infty} V^{(j)}$, $\hat{F} = F/N$ endowed with the quotient topology $\tau' = \tau^*_q$ is metrizable.

If $K : F \to \hat{F}$ is the quotient map and each $I_n : E_n \to F$ is the canonical injection, then the collection $\{\hat{I}_n\}_{n=1}^{\infty}$, where each $\hat{I}_n = K \circ I_n$, is equicontinuous and it can be easily seen that $U^{(0)}$ is a neighbourhood of 0 in E. #

The following is a partial converse of 3.24.

3.26 LEMMA If $E[\tau]$ is a topological vector space, and if for each metrizable topological vector space $F[\tau']$, $L_b(E,F)$ is metrizable, then $E[\tau]$ satisfies property 3.3 (a).

PROOF. We construct a metrizable space $\hat{F}[\tau']$. Let $\{V_\gamma^{(0)} : \gamma \in \Gamma\}$ be a neighbourhood basis of 0 for $E[\tau]$, consisting of circled neighbourhoods of 0, and for each $\gamma \in \Gamma$, let $\{V_\gamma^{(j)}\}_{j=1}^{\infty}$ be a defining sequence for $V_\gamma^{(0)}$ consisting of circled neighbourhoods of 0. Let $F = \bigoplus_{\gamma \in \Gamma} E_\gamma$ be the algebraic direct sum of the linear spaces $\{E_\gamma\}_{\gamma \in \Gamma}$ where each $E_\gamma = E$. The sets $V^{(j)} = \bigoplus_{\gamma \in \Gamma} V_\gamma^{(j)}$, $j = 0,1, \ldots$ form a neighbourhood basis for a pseudo-metrizable vector topology τ^* on F. Letting $N = \bigcap_{j=0}^{\infty} V^{(j)}$, $\hat{F} = F/N$ endowed with the quotient topology $\tau' = \tau^*_q$ is metrizable.

Obviously, $L(E,F)$ contains the canonical injections $I_\gamma : E_\gamma \to F$ for each $\gamma \in \Gamma$. Letting $\hat{I}_\gamma = K \circ I_\gamma$ where K is the canonical surjection $K : F \to \hat{F}$, then $\hat{I}_\gamma \in L(E, \hat{F})$ for each $\gamma \in \Gamma$. We let $\hat{V}^{(j)} = K[V^{(j)}]$ for each $j \geq 0$.

We now show that for all bounded $B \subset E$, the sets $\langle B,\hat{V}^{(j)}\rangle^{-1}[\hat{V}^{(i)}] = \bigcap_{f\in\langle B,\hat{V}^{(j)}\rangle} f^{-1}[\hat{V}^{(i)}]$ for all $j,i \geq 0$, are bounded in E. Since $\hat{V}^{(i+1)} \subset \hat{V}^{(i)}$ for $i \geq 0$, it follows that $\langle B,\hat{V}^{(j)}\rangle^{-1}[\hat{V}^{(i+1)}] \subset \langle B,\hat{V}^{(j)}\rangle^{-1}[\hat{V}^{(i)}]$, so it suffices to show that $\langle B,\hat{V}^{(j)}\rangle^{-1}[\hat{V}^{(0)}]$ is bounded in E for each $j \geq 0$.

Because $\langle B,\hat{V}^{(j)}\rangle$ is a neighbourhood of 0 in $L_b(E,\hat{F})$, for each $\hat{I}_\gamma$ there exists $\lambda_\gamma > 0$, such that $\lambda_\gamma \hat{I}_\gamma \in \langle B,\hat{V}^{(j)}\rangle$. For each $\gamma \in \Gamma$,

$$\begin{aligned}\langle B,\hat{V}^{(j)}\rangle^{-1}[\hat{V}^{(0)}] &\subset (\lambda_\gamma \hat{I}_\gamma)^{-1}[\hat{V}^{(0)}]\\ &= \frac{1}{\lambda_\gamma} I_\gamma^{-1}[K^{-1}[\hat{V}^{(0)}]]\\ &= \frac{1}{\lambda_\gamma} I_\gamma^{-1}(V^{(0)} + N)\\ &\subset \frac{1}{\lambda_\gamma} I_\gamma^{-1}(V^{(0)} + V^{(0)})\\ &\subset \frac{1}{\lambda_\gamma}(V_\gamma^{(0)} + V_\gamma^{(0)}).\end{aligned}$$

As $\{V_\gamma^{(0)}\}_{\gamma\in\Gamma}$ forms a neighbourhood basis of 0 in $E[\tau]$ so does $\{V_\gamma^{(0)} + V_\gamma^{(0)}\}_{\gamma\in\Gamma}$. Thus we have shown that the above-mentioned sets are bounded in $E[\tau]$.

By hypothesis there exists a countable neighbourhood basis $\{W_n\}_{n\in\mathbb{N}}$ of 0 in $L_b(E,\hat{F})$ such that $W_{n+1} + W_{n+1} \subset W_n$ for all $n \geq 1$. The sets $\langle B,\hat{V}^{(j)}\rangle$, where B is bounded in E and $j \geq 0$, form a neighbourhood basis of 0 in $L_b(E,\hat{F})$, so by what we have shown above the sets

$C_n^{(j)} = W_n^{-1}[\hat{V}^{(j)}]$, $n \geq 1$, $j \geq 0$ are all bounded in E.

If B is an arbitrary bounded set in E, then $\langle B,\hat{V}^{(j)}\rangle$ is a neighbourhood of 0 in $L_b(E,\hat{F})$, so there exists $W_n \subset \langle B,\hat{V}^{(j)}\rangle$. Then we have

$$B \subset \langle B, \hat{V}^{(j)} \rangle^{-1}[\hat{V}^{(j)}] \subset W_n^{-1}[\hat{V}^{(j)}] = C_n^{(j)} \subset C_n^{(0)}.$$

Because the $C_n^{(0)}$ are countable, there exists a fundamental sequence of circled closed bounded sets. #

Results 3.22, 3.24, 3.25 and 3.26 together yield the following characterization of (UDF)-spaces.

3.27 THEOREM A Hausdorff topological vector space $E[\tau]$ is a (UDF)-space if and only if for all metrizable spaces $F[\tau']$

a) $L_b(E,F)$ is metrizable, and

b) if $A \subset L_b(E,F)$ is bounded and A is a union of a sequence of equicontinuous subsets of $L(E,F)$, then A is equicontinuous. #

11 Permanence properties of (UDF)-spaces

We now show that the class of (UDF)-spaces possesses many permanence properties similar to those of (DF)-spaces. We begin with the quotient structure.

3.28 THEOREM. If $E[\tau]$ is a (UDF)-space and $H \subset E$ a closed subspace, then the quotient space $E/H[\tau_q]$ is a (UDF)-space.

PROOF. It is first verified that $E/H[\tau_q]$ has property 3.3(b). Let $K : E \to E/H$ be the canonical quotient mapping, and for each n, $\hat{B}_n = \overline{K[B_n]}$ where $\{B_n\}_{n=1}^{\infty}$ is a fundamental sequence of circled bounded sets in E such that $B_n + B_n \subset B_{n+1}$ for each n. Obviously $\{\overline{K[B_n]}\}_{n=1}^{\infty}$ is a sequence of closed circled bounded sets in E/H. We show that if $\{\hat{U}_n^{(0)}\}_{n=1}^{\infty}$ is a sequence of circled closed neighbourhoods of 0 in E/H and if for each n, $\hat{U}_n^{(0)}$ has a defining sequence $\{\hat{U}_n^{(j)}\}_{j=1}^{\infty}$ such that for each j,

$\hat{U}^{(j)} = \bigcap_{n=1}^{\infty} \hat{U}_n^{(j)}$ absorbs $\hat{B}_n$, then $\hat{U}^{(0)}$ is a neighbourhood of 0 in E/H.

For $n \geq 1$ and $j \geq 0$, let $U_n^{(j)} = K^{-1}[\hat{U}_n^{(j)}]$. Then $U_n^{(0)}$ is a circled closed neighbourhood of 0 in E for each n, and $\{U_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $U_n^{(0)}$. Since $\hat{U}^{(j)}$ absorbs all $\hat{B}_m$, there exists for each $\hat{B}_m$ a number $\lambda > 0$ with $\lambda\hat{B}_m \subset \hat{U}^{(j)} \subset \hat{U}_n^{(j)}$ for all $n \geq 1$. We have $\lambda B_m \subset K^{-1}[\lambda\hat{B}_m] \subset K^{-1}[\hat{U}_n^{(j)}] = U_n^{(j)}$, for each n. So, as a result, $U^{(j)} = \bigcap_{n=1}^{\infty} U_n^{(j)}$ absorbs all bounded sets in E, and thus $U^{(0)}$ is a neighbourhood of 0 in E. Now it follows that

$$K[U^{(0)}] = K[\bigcap_{n=1}^{\infty} U_n^{(0)}] \subset \bigcap_{n=1}^{\infty} K[U_n^{(0)}]$$

$$= \bigcap_{n=1}^{\infty} \hat{U}_n^{(0)} = \hat{U}^{(0)},$$ hence $\hat{U}^{(0)}$ is a neighbourhood of 0 in E/H.

~~To complete the proof we show that every bounded set in $E/H[\tau_q]$ is a subset of some $\overline{K[B_n]}$, i.e. $\{\overline{K[B_n]}\}_{n=1}^{\infty}$ is a fundamental sequence of bounded sets in E/H.~~

Suppose $\{\hat{B}_n\}_{n=1}^{\infty}$ is not a fundamental sequence. Then there is some bounded set $\hat{B}$ which is not a subset of any $\hat{B}_n$. If $\hat{B}$ is absorbed by some $\hat{B}_n$ it will follow that $\hat{B}$ is contained in some set in the sequence. Hence it may be assumed that $\hat{B}$ is not absorbed by any $\hat{B}_n$.

For each $n \geq 1$ choose $x_n \in \frac{1}{n}\hat{B}$ such that $x_n \notin \hat{B}_n$. Then for each n there is a closed circled neighbourhood $\hat{U}_n^{(0)}$ of 0 in E/H with a defining sequence of closed circled neighbourhoods $\{\hat{U}_n^{(j)}\}_{j=1}^{\infty}$ of 0 such that $x_n \notin \overline{\hat{B}_n + \hat{U}_n^{(0)}}$.

Let

$$\hat{W}_n^{(j)} = \begin{cases} \overline{\hat{B}_{n-j} + \hat{U}_n^{(j)}} & j = 0, 1, \ldots, n-1 \\ \hat{U}_n^{(j)} & j \geq n. \end{cases}$$

Then $\hat{W}_n^{(j)}$ is a closed circled neighbourhood of 0 in E/H for $n \geq 1$ and $j \geq 0$, and because $\hat{B}_n + \hat{B}_n \subset \hat{B}_{n+1}$ for each n, we have $\{\hat{W}_n^{(j)}\}_{j=1}^{\infty}$ is a defining sequence for $\hat{W}_n^{(0)}$.

Consider the sets $\hat{W}^{(j)} = \bigcap_{n=1}^{\infty} \hat{W}_n^{(j)}$. Let $\hat{B}_m$ be a member of the sequence of bounded sets. If $n \geq m + j$ we obviously have $\hat{B}_m \subset \hat{W}_n^{(j)}$, so $\hat{B}_m \subset \bigcap_{n=m+j}^{\infty} \hat{W}_n^{(j)}$, and also $\bigcap_{n=1}^{m+j-1} \hat{W}_n^{(j)}$ is a neighbourhood of 0 in E/H, consequently $\hat{W}^{(j)} = (\bigcap_{n=m+j}^{\infty} \hat{W}_n^{(j)}) \cap (\bigcap_{n=1}^{m+j-1} \hat{W}_n^{(j)})$ absorbs $\hat{B}_m$. By the first part of the proof, $\hat{W}^{(0)}$ is a neighbourhood of 0 in E/H and also we have $x_n \notin \hat{W}^{(0)}$ for each n. But $\{x_n\}_{n=1}^{\infty}$ is a null sequence in E/H which is a contradiction. This completes the proof. #

Now we deal with the completions of (UDF)-spaces.

3.29 THEOREM If $E[\tau]$ is a (UDF)-space, then

a) the completion $\tilde{E}[\tilde{\tau}]$ of $E[\tau]$ is a (UDF)-space, and

b) $E[\tau]$ is complete if and only if it is quasi-complete.

PROOF The proof of (a) is analogous to the proof of 3.28. From this we obtain that if $\{B_n\}_{n=1}^{\infty}$ is a fundamental sequence of closed circled bounded sets in $E[\tau]$, then the sequence $\{\tilde{B}_n\}_{n=1}^{\infty}$ consisting of the closures in $\tilde{E}[\tilde{\tau}]$ of the B_n is a fundamental sequence of circled closed bounded sets in $\tilde{E}[\tilde{\tau}]$. We use this fact to prove (b).

If $x \in \tilde{E}$, then $x \in \tilde{B}_n$ for some n. But B_n is closed and bounded in $E[\tau]$, so B_n is complete in $E[\tau]$, and thus complete in $\tilde{E}[\tilde{\tau}]$. Hence $\tilde{B}_n = B_n$ and $x \in E$. Thus $E = \tilde{E}$, and consequently $E[\tau]$ is complete. #

By IYAHEN [10], if $E[\tau]$ is the (*)-inductive limit of the spaces

$\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ by the maps $\{u_\alpha\}_{\alpha\in A}$ and for each $\alpha \in A$, V_α is a circled neighbourhood of 0 in E_α, then $U = \bigcup_\phi \sum_{\alpha\in\phi} u_\alpha[V_\alpha]$ where the union is taken over all finite subsets ϕ of A is a neighbourhood of 0 in $E[\tau]$. If A is countable, then as V_α runs through a basis of circled neighbourhoods of 0 in E_α, the collection of U obtained forms a base of neighbourhoods of 0 in $E[\tau]$. In particular if the $E_\alpha[\tau_\alpha]$ are locally convex and A is countable, $E[\tau]$ is locally convex because each U may be written as the union of an increasing sequence of absolutely convex sets provided we choose suitable neighbourhood bases of 0 in the spaces $E_\alpha[\tau_\alpha]$. Consequently, in this instance, the (*)-inductive limit of the spaces $\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ by the mappings $\{u_\alpha\}_{\alpha\in A}$, if separated, coincides with the inductive limit of the spaces.

The notion of the (*)-direct sum of a class of topological vector spaces as a special case of the (*)-inductive limit was introduced in [10]. Let $\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ be a class of topological vector spaces, $E = \bigoplus_{\alpha\in A} E_\alpha$ the algebraic direct sum, and $\{I_\alpha\}_{\alpha\in A}$ the class of injection mappings $I_\alpha : E_\alpha \to E$. Then the (*)-inductive limit of the spaces $\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ by the maps $\{I_\alpha\}_{\alpha\in A}$ is called the (*)-*direct sum* $\bigoplus E_\alpha[\tau_\alpha]$ of the spaces $\{E_\alpha\}_{\alpha\in A}$.

3.30 THEOREM. The (*)-direct sum of a sequence $\{E_n[\tau_n]\}_{n=1}^\infty$ of (UDF)-spaces is again a (UDF)-space.

PROOF Let $E[\tau]$ be the (*)-direct sum of the spaces $\{E_n[\tau_n]\}_{n=1}^\infty$. IYAHEN [10] showed that if B is bounded in $E[\tau]$, B is contained in a sum of bounded subsets of finitely many of the E_n. Hence, if $\{B_n^{(j)}\}_{j=1}^\infty$ is a fundamental sequence of bounded sets in E_n, for each n, then $B_n = \bigoplus_{i=1}^n B_i^{(n)}$ is a fundamental sequence of bounded sets in $E[\tau]$.

Because τ is Hausdorff, we need only to show that property 3.3(b) is satisfied. Let $\{W_n^{(0)}\}_{n=1}^{\infty}$ be a sequence of circled closed neighbourhoods of 0 in E, and for each n, let $\{W_n^{(j)}\}_{j=1}^{\infty}$ be a defining sequence for $W_n^{(0)}$ such that for each $j \geq 0$, the set $W^{(j)} = \bigcap_{n=1}^{\infty} W_n^{(j)}$ absorbs all bounded sets. For each m let $I_m : E_m \to E$ be the canonical injection.

Then the $I_m^{-1}[W_n^{(0)}]$ are circled closed neighbourhoods of 0, and the sets $\{I_m^{-1}[W_n^{(j)}]\}_{j=1}^{\infty}$ form a defining sequence for $I_m^{-1}[W_n^{(0)}]$ for each n.

Since $I_m^{-1}[W^{(j)}] = \bigcap_{n=1}^{\infty} I_m^{-1}[W_n^{(j)}]$ and I_m is continuous, we have $I_m^{-1}[W^{(j)}]$ absorbs all bounded sets of E_m for each j; consequently, $I_m^{-1}[W^{(0)}]$ is a neighbourhood of 0 in E_m for each m.

It can be shown that $W^{(0)}$ is a neighbourhood of 0 in $E[\tau]$, so that the proof is complete. #

In [10] it is further shown that the (*)-inductive limit of a class of spaces by a class of mappings is topologically isomorphic to a quotient of the (*)-direct sum of the spaces. So from 3.28 and 3.30 we have

3.31 THEOREM A Hausdorff (*)-inductive limit of a countable number of (UDF)-spaces is a (UDF)-space. #

In section 8 we remarked that locally bounded spaces are (UDF)-spaces. Using 3.31 we can construct (UDF)-spaces by considering Hausdorff (*)-inductive limits of the spaces $\{\ell^{\frac{1}{n}}\}_{n=1}^{\infty}$.

With the help of theorem 3.31 we can show that many familiar (DF)-spaces are (UDF)-spaces. Each bornological (DF)-space is the inductive limit of a sequence of normed spaces. Thus we have

3.32 THEOREM Every locally convex bornological (DF)- space is a (UDF)-space. #

Similar to inductive limits (= locally convex hulls) we can write the (*)-inductive limit $E[\tau]$ of the spaces $\{E_\alpha[\tau_\alpha]\}_{\alpha\in A}$ by the mappings $\{A_\alpha\}_{\alpha\in A}$ as $E[\tau] = \sum_{\alpha\in A} A_\alpha[E_\alpha[\tau_\alpha]]$.

3.33 THEOREM Let $\{E_n[\tau_n]\}$ be a sequence of (UDF)-spaces, $\{A_n\}_{n=1}^{\infty}$ be linear maps $A_n : E_n \to E$ where $E[\tau] = \sum_{n=1}^{\infty} A_n[E_n[\tau_n]]$ is Hausdorff. Then each bounded set B of E lies in the closure of the sum of a finite number of bounded sets $B_{n_i} \subset E_{n_i}$ $(i = 1,2, \ldots, m)$, i.e. $B \subset \overline{\sum_{i=1}^{m} A_{n_i}[B_{n_i}]}$.

PROOF: Define a linear map

$$A : \bigoplus_{n=1}^{\infty} E_n[\tau_n] \to \sum_{n=1}^{\infty} A_n[E_n[\tau_n]] \quad \text{by}$$

$A(\bigoplus_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} A_n(x_n)$. Obviously A is well-defined and linear. Let $N(A)$ be the null space of A. Consider the quotient space $\bigoplus_{n=1}^{\infty} E_n[\tau_n]/N(A)$.

The mapping

$$\hat{A} : \bigoplus_{n=1}^{\infty} E_n[\tau_n]/N(A) \to \sum_{n=1}^{\infty} A_n[E_n[\tau_n]] \quad \text{defined by}$$

$\hat{A}([\bigoplus_{n=1}^{\infty} x_n]) = \sum_{n=1}^{\infty} A_n(x_n)$ is an isomorphism. Let

$K : \bigoplus_{n=1}^{\infty} E_n[\tau_n] \to \bigoplus_{n=1}^{\infty} E_n[\tau_n]/N(A)$ be the canonical quotient map; then $A = \hat{A} \circ K$.

As mentioned earlier, when V_n runs through a basis of circled neighbourhoods of 0 in $E_n[\tau_n]$, then the sets of the form

$U = \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} I_n[V_n]$ and $A[U] = \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} A_n[V_n]$ form bases of neighbourhoods of 0 for $\bigoplus_{n=1}^{\infty} E_n[\tau_n]$ and $\sum_{n=1}^{\infty} A_n[E_n[\tau_n]]$ respectively. It can easily be verified that A is a continuous and open mapping; hence by [13; 15,4.(4)] $\hat{A}$ is a topological isomorphism.

If B is bounded in $E[\tau]$, then $\hat{B} = \hat{A}^{-1}[B]$ is bounded in $\bigoplus_{n=1}^{\infty} E_n[\tau_n]/N(A)$. As in 3.28, there is a bounded set B^* in the (*)-direct sum, such that $\hat{B} \subset \overline{K[B^*]}$. So there is a finite number of bounded sets $B_{n_i} \subset E_{n_i}$ $(i = 1, \ldots, m)$ so that

$$B^* \subset \bigoplus_{i=1}^{m} B_{n_i}.$$

So, $\hat{B} \subset \overline{K[\bigoplus_{i=1}^{m} B_{n_i}]}$ and then

$B = \hat{A}[\hat{A}^{-1}[B]] \subset \hat{A}[\overline{K[\bigoplus_{i=1}^{m} B_{n_i}]}]$. Since $\hat{A}$ is a topological isomorphism

$$\hat{A}[\overline{K[\bigoplus_{i=1}^{m} B_{n_i}]}] = \overline{\hat{A}[K[\bigoplus_{i=1}^{m} B_{n_i}]]}, \text{ so}$$

$$B \subset \overline{\hat{A}[K[\bigoplus_{i=1}^{m} B_{n_i}]]} = \overline{A[\bigoplus_{i=1}^{m} B_{n_i}]} = \overline{\sum_{i=1}^{m} A_{n_i}[B_{n_i}]}. \quad \#$$

CHAPTER IV

SCHWARTZ SPACES

This chapter mainly concerns itself with a certain class of locally convex spaces defined and called Schwartz spaces by GROTHENDIECK [7]. In a general setting, one can consider Schwartz spaces as a special case of those locally convex spaces in which every bounded set is precompact, or as a more general case of nuclear spaces. The theories of Schwartz spaces and nuclear spaces run parallel, and many of the properties could be proved simultaneously. Since the definition itself leans on the properties of transpose maps, we begin by reviewing basic results in this area. We then proceed to define and give various characterizations of Schwartz spaces. A general representation theorem for precompact semi-norms on a locally convex space is proved using the Banach space c_0. The concept of local convergence plays an important role in this discussion. Using this, several characterizations of Schwartz spaces are obtained based on the works of RANDTKE [14] and TERZIOĞLU [19]. The theory runs somewhat similar to that of nuclear spaces.

We conclude the chapter with a discussion of some hereditary properties of Schwartz spaces. In essence they form a "variety" ([5]), in the sense that they are closed under subspaces (not necessarily closed), separated quotients, arbitrary products, and isomorphic images.

12 Transpose mappings in normed spaces

Let $E[\tau]$ and $F[\tau']$ be locally convex spaces and $t : E \to F$ be a linear map. The transpose map $t' : F' \to E^*$ is defined by $f \mapsto f \circ t$ for f in F'. Obviously t' is linear. The following results are well-known. For proofs we refer to ROBERTSON and ROBERTSON [16] or HORVÁTH [9].

4.1 THEOREM

a) If t is continuous, then t is $\tau_s(E') - \tau_s(F')$ continuous.

b) $t'[F'] \subset E'$ if and only if t is weakly $(\tau_s(E') - \tau_s(F'))$ continuous; in this case, t' is weakly $(\tau_s(F) - \tau_s(E))$ as well as strongly $(\tau_b(F) - \tau_b(E))$ continuous.

c) If t is weakly continuous, the transpose t'' of t' from E'' into F'' is well-defined, and further $t''|E = t : E \to F$. #

4.2 DEFINITION The linear map t between locally convex spaces $E[\tau]$ and $F[\tau']$ is said to be *compact*, (*precompact*, *bounded*) if there is a neighbourhood U of 0 in E such that $t[U]$ is relatively compact, (precompact, bounded) in F. #

Obviously compact maps are precompact, and precompact maps are bounded. The following characterization of precompact maps between normed spaces by means of the transpose map is often called Schauder's Theorem.

4.3 THEOREM Let E and F be normed spaces and $t : E \to F$ be linear. Then t is precompact if and only if $t' : F' \to E'$ is a compact mapping of the norm duals.

PROOF Let T be the image in F of the unit ball S of E. Then $t'[T^o] \subset S^o$; so that t' is continuous if we equip F' with the topology $\tau_c(F)$ and E' with the topology $\tau_b(E)$, i.e. the norm topology. The closed unit ball B of F' is the polar of the unit ball in F, and hence is equicontinuous. Since $\tau_c(F)$ induces $\tau_s(F)$ on equicontinuous sets, B is $\tau_c(F)$-compact. Consequently the image of B under t' is compact in the norm topology on E'.

The converse follows similarly from theorem 4.1. #

Let $E[\tau]$ be a locally convex space and E' its dual. Let $M \subset E$ be a closed subspace and $\phi : E \to E/M$ the canonical surjection. Then it is easy to see that the transpose $\phi' : (E/M)' \to E'$ defined by $f \mapsto f \circ \phi$ is injective with image $M^{\perp}$; so, we may consider the restriction $\phi' : (E/M)' \to M^{\perp}$. It can easily be shown that ϕ' is a topological isomorphism when we equip $(E/M)'$ with the topology $\tau_s(E/M)$ and $M^{\perp}$ with the topology $\tau_s(E)$ restricted to $M^{\perp}$; so we identify $M^{\perp}$ with $(E/M)'$.

By 4.1, $\phi' : (E/M)'[\tau_b(E/M)] \to E'[\tau_b(E)]$ is continuous; hence, $\tau_b(E)|M^{\perp} \leq \tau_b(E/M)$. We now give a condition under which the reverse inequality holds.

4.4 LEMMA Let $E[\tau]$ be a locally convex space with dual E', and $M \subset E$ a closed subspace. Then, $\tau_b(E)|M^{\perp} = \tau_b(E/M)$ if and only if for every bounded set B of $E/M[\tau_q]$, there exists a bounded set B_1 of E such that the closed absolutely convex hull of the canonical image of B_1 in E/M contains B.

PROOF We need only show $\tau_b(E/M) \leq \tau_b(E)|M^{\perp}$. A basis of neighbourhoods of 0 in $\tau_b(E/M)$ is given by the family of sets B^o where B is closed absolutely convex and bounded in $E/M[\tau_q]$. A basis of neighbourhoods of 0 in $\tau_b(E)|M^{\perp}$ is given by the sets $B_1^{\,o} \cap M^{\perp}$ where B_1 is bounded and absolutely convex in E. So $\tau_b(E/M) \leq \tau_b(E)|M^{\perp}$ if and only if for each closed absolutely convex bounded set $B \subset E/M$, there exists an absolutely convex bounded $B_1 \subset E$ such that $B^0 \supset B_1^{\,0} \cap M^{\perp} = (B_1 \cup M)^0 = (\langle B_1 \cup M \rangle)^0$. But $\langle B_1 \cup M \rangle = B_1 + M$ since B_1 is balanced and M is a subspace. Our condition is satisfied if and only if $B^0 \supset (B_1 + M)^0$. By polarity it immediately follows that B is contained in the closed absolutely convex hull of the canonical image of B_1. #

This result will be needed when we consider permanence properties of Schwartz spaces.

Again let $E[\tau]$ be locally convex, and $M \subset E$ be a linear subspace. Let $j : M \to E$ be the canonical injection. Then the transpose $j' : E' \to M'$ (where $M' = (M[\tau|M])'$) is defined by $f \mapsto f \circ j$, and is obviously onto. The kernel of j is $M^{\perp}$.

Consider the quotient space $E'/M^{\perp}$. Then the linear map $t : E'/M^{\perp} \to M'$ defined by $f(M^{\perp}) \mapsto f|M$ is a bijection. Since $j' : E'[\tau_s(E)] \to M'[\tau_s(M)]$ is continuous t is continuous. By HORVÁTH [9; Ch. 3, Prop. 13.2], t is a topological isomorphism if M is closed. So we identify M' with $E'/M^{\perp}$ (even if M is not closed). Letting $\tau_b(E)_q$ represent the quotient topology of quotient space $E'[\tau_b(E)]$ by the subspace $M^{\perp}$ we obtain from theorem 4.1 that $t : E'/M^{\perp}[\tau_b(E)_q] \to M'[\tau_b(M)]$ is continuous, thus $\tau_b(M) \leq \tau_b(E)_q$. A condition for equality is given in the following result.

4.5 LEMMA Let $E[\tau]$ be a locally convex space, and M a vector subspace of E. Then the quotient topology $\tau_b(E)_q$ on $E'/M^{\perp}$ is equivalent to $\tau_b(M)$ if and only if for every bounded set B of E there exists a bounded set B_1 of M (in the restricted topology of τ to M) such that every continuous linear form on M bounded by 1 on B_1 can be extended to a continuous linear form on E bounded by 1 on B.

PROOF We need only show that the condition is equivalent to the fact that $\tau_b(E)_q \leq \tau_b(M)$. A fundamental system of neighbourhoods of the former topology is given by the canonical images of the B^o where B is bounded in E. A fundamental system of neighbourhoods in the latter is given by the $B_1{}^o$ where B_1 is bounded in M. So $\tau_b(E)_q \leq \tau_b(M)$ if and only if for each B, there is a B_1 such that $B_1{}^o \subset K[B^o]$ where $K : E' \to E'/M^\perp$ is the canonical surjection.

We now prove the equivalence of this and the condition in the statement. Let $h \in M'$ and h be bounded by 1 on B_1. Then $h = f|M$ for some $f \in E'$. Since $B_1{}^o \subset K[B^o]$, $f(M^\perp) \in K[B^o]$. Consequently for some $m \in M^\perp$, $f + m \in B^o$. Since $m \in M^\perp$, $f + m$ extends h.

Conversely, let B be bounded in E and B_1 bounded in M such that the condition in the statement is satisfied. Let $f \in B_1{}^o$; then $f \in M'$ such that f is bounded by 1 on B_1. Then, letting f_1 be an extension of f to E such that $f_1 \in B^o$, we have $f = K(f_1)$ since $f_1|M = f$. Thus, $B_1{}^o \subset K[B^o]$ and the proof is complete. #

We now give a class of spaces which satisfies the condition of lemma 4.5.

4.6 THEOREM Let $E[\tau]$ be a semi-reflexive space with dual E', and $M \subset E$ a closed linear subspace. Then

a) M is semi-reflexive, and

b) the strong topology $\tau_b(M)$ on $E'/M^\perp$ coincides with the quotient topology $\tau_b(E)_q$ on $E'/M^\perp$.

PROOF It can trivially be shown that $\tau_s(M') = \tau_s(E')|M$. So if B is bounded in M, B is bounded in E; hence B is weakly relatively compact in E, since E is semi-reflexive. Because M is closed, B is relatively $\tau_s(M')$-compact. Thus $M[\tau|M]$ is semi-reflexive, proving (a).

We shall use 4.5 to prove (b). Let B be closed absolutely convex and bounded in $E[\tau]$ and $B_1 = 2(B \cap M)$. Let $f \in M'$ such that f is bounded by 1 on B_1 and so by $1/2$ on $B \cap M$. Let $G = f^{-1}[\{1\}]$. Then $G \cap B \cap M = \phi$, and G is closed in E, since G is closed in M, and M is closed in E. Obviously, G is convex hence G is weakly closed in E, and $E \sim G$ is weakly open.

If $b \in B \cap M$, then $b \notin G$, and if $b \in B \cap (E \sim M)$, then $b \notin M$, so that $B \subset E \sim G$. Because B is bounded, B is $\tau_s(E')$-compact. By SCHWARTZ [18; 2.8], there exists a weakly open absolutely convex neighbourhood U of 0 such that $B + U \subset E \sim G$. Hence $(B + U) \cap G = \phi$.

Let $x \in M$ such that $f(x) = 1$, and $Q = f^{-1}[\{0\}]$. Then $G = x + Q$, and $(B + U - x) \cap Q = \phi$, where Q is weakly closed, and $B + U - x$ is weakly open and convex in E. By the Hahn-Banach Theorem, there exists a weakly closed hyperspace H containing Q such that $(B + U - x) \cap H = \phi$.

Let f' be the linear functional associated with H such that $f'(x) = 1$. Since H is weakly closed, f' is continuous. If $y \in M$, $y = \lambda x + q$ where $\lambda \in K$, and $q \in Q$; thus

$$f'(y) = f'(\lambda x + q) = \lambda = f(\lambda x + q) = f(y).$$ So

f' extends f.

If there exists $b \in B$ such that $|f'(b)| > 1$, then since B is circled we can assume $f'(b) = 1$. Hence $B \cap (x + H) \neq \phi$, a contradiction. #

The next result will be useful in finding a class of Schwartz spaces which satisfy lemma 4.4.

4.7 LEMMA Let E and F be (F)-spaces and $f : E \to F$ a continuous and surjective linear mapping. For every compact $K \subset F$, there exists compact $H \subsetneq E$ such that $f[H] = K$. Further, if K is absolutely convex we can find an absolutely convex H.

PROOF By the open mapping theorem, f is open. Let $d : E \times E \to \mathbb{R}^+$ be a translation-invariant metric which generates the topology on E; then (E,d) is a complete metric space.

For $n = 1,2, \ldots$ let $\mathcal{B}_n$ be the class of open balls in E with radius $1/2^n$. Letting B run through the sets in $\mathcal{B}_1$, the class of sets $f[B]$ form an open covering of K. Choose a finite set $H_1 = \{x_1, \ldots, x_m\}$ of points of E such that if B_i for $1 \leq i \leq m$ is the open ball in $\mathcal{B}_1$ centered on x_i, then $\bigcup_{i=1}^{m} f[B_i]$ covers K.

Assume for the indices $1 \leq i \leq n$ we have constructed a finite subset H_i of E with the following two properties

1) $H_i \subset H_{i+1}$, and each point of H_{i+1} is at a distance $< 1/2^i$ from H_i $(1 \leq i \leq n - 1)$, and

2) if B runs through the open balls in $\mathcal{B}_i$ centered at a point of H_i, then the union of the sets $f[B]$ covers K.

Let $\mathcal{B}'_{n+1}$ be the collection of sets in $\mathcal{B}_{n+1}$, whose centre x satisfies $d(x, H_n) < 1/2^n$. If $y \in K$, then $y \in f[B]$ where B is an open ball of radius $1/2^n$ centered on some $h \in H_n$. So there exists $x \in E$ for which $f(x) = y$ and $d(x,h) < 1/2^n$. Hence as B runs through the balls of $\mathcal{B}'_{n+1}$, $\bigcup f[B]$ covers K. Choose a finite set

$G_{n+1} \subset E$ such that as B runs through the open balls in E with radius $1/2^{n+1}$ and centre at a point of G_{n+1}, then the sets $f[B]$ cover K. Set $H_{n+1} = H_n \cup G_{n+1}$. Obviously H_{n+1} satisfies properties (1) and (2). Consequently for each n there is such a set.

Let $L = \bigcup_{n=1}^{\infty} H_n$. We now prove L is precompact. Let $n \geq 1$. If $y_{n+p} \in H_{n+p}$, where $p > 0$, there is a finite sequence of points $y_{n+i} \in H_{n+i}$, $0 \leq i < p$, such that

$$d(y_{n+i}, y_{n+i+1}) < 1/2^{n+i}. \text{ Thus}$$

$$d(y_n, y_{n+p}) < \sum_{i=0}^{p-1} \frac{1}{2^{n+i}} < 1/2^{n-1}. \text{ Obviously } L$$

is precompact.

Since (E,d) is a complete metric space $\bar{L}$ is compact. We now show $K \subset f[\bar{L}]$. Let $z \in K$, then

$$d(f^{-1}[\{z\}], H_n) < 1/2^n \text{ for every } n; \text{ so}$$

$d(f^{-1}[\{z\}], \bar{L}) = 0$. Because the function $q : E \to \mathbb{R}$ defined by $x \mapsto d(f^{-1}[\{z\}], x)$ is continuous, we have $q[\bar{L}]$ is compact in $\mathbb{R}$; consequently $q[\bar{L}]$ contains its infimum. Thus, there exists $x \in \bar{L}$ such that $d(f^{-1}[\{z\}], x) = 0$. Hence $x \in \overline{f^{-1}[\{z\}]} = f^{-1}[\{z\}]$ since $\{z\}$ is a closed set and f is continuous. We have $\bar{L} \cap f^{-1}[\{z\}] \neq \phi$, and $K \subset f[\bar{L}]$. Letting $H = \bar{L} \cap f^{-1}[K]$, then $f[H] = K$, and H is compact.

If K is absolutely convex, choose L as above. Then $\overline{\Gamma L}$ is compact and $K \subset f[\overline{\Gamma L}]$. Letting $H = \overline{\Gamma L} \cap f^{-1}[K]$, H is absolutely convex and compact, and $f[H] = K$. #

Let $E[\tau]$ be a locally convex space, and $\mathcal{U}(E)$ and $\mathcal{B}(E)$ be the classes of closed absolutely convex neighbourhoods of 0 and of the closed absolutely convex bounded sets of E, respectively. If p is a semi-norm on

E, we denote by E_p, the space E with the semi-norm topology induced by p.

We let $\hat{E}_p = E/\{x \in E : p(x) = 0\}$ with norm defined by $||x(P)|| = p(x)$ where $P = \{x \in E : p(x) = 0\}$. If $K_p : E \to \hat{E}_p$ denotes the canonical surjection, K_p is continuous if and only if p is continuous. There is an obvious correspondence between the continuous semi-norms on E and the elements of $U(E)$. Similarly for $U \in U(E)$ we let p_U be the gauge of U, and set $E_U = \hat{E}_{p_U}$ and $K_U = K_{p_U}$. If $V \in U(E)$ and $V \subset U$, we have a well-defined linear map $K_U^V : E_V \to E_U$ given by $x(V) \mapsto x(U)$ where $x(V)$ and $x(U)$ represent the obvious equivalence classes. Since $V \subset U$, $p_U \leq p_V$ so that K_U^V is continuous.

We let $\tilde{E}_U$ denote the completion of E_U where $U \in U(E)$. Let $\tilde{K}_U : E \to \tilde{E}_U$ denote the composition of the maps $K_U : E \to E_U$ and the identity $i : E_U \to \tilde{E}_U$. From above we see that $K_U = K_U^V \circ K_V$. Let $\tilde{K}_U^V : \tilde{E}_V \to \tilde{E}_U$ be the uniquely determined extension. Clearly $\tilde{K}_U = \tilde{K}_U^V \circ \tilde{K}_V$.

If $A \in B(E)$, let $E(A) = \bigcup_{n=1}^{\infty} nA$ be the space semi-normed by the gauge p_A on the set A. Since A is bounded, p_A is a norm, and since A is closed in $E[\tau]$, it follows that A is the closed unit ball of $E(A)$. If $A \subset B \in B(E)$, we denote by $I_B^A : E(A) \to E(B)$, the canonical injection. Since $A \subset B$, I_B^A is continuous.

It is easy to see that the mapping $I : (E_U)' \to E'(U^o)$ defined by $f \mapsto f \circ K_U$ is a bijection and an isometry if the spaces have their norm topologies. So we can consider the space $E'(U^o)$, a Banach space, to be the strong dual of the normed space E_U.

As a result of this, the transpose of K_U^V is the injection $I_{V^o}^{U^o} : E'(U^o) \to E'(V^o)$.

If $A \in \mathcal{B}(E)$, then $A^o \in \mathcal{U}(E')$ for the topology $\tau_b(E)$ on E'. Thus the space E'_{A^o} may be constructed. Consider the mapping $J : E'_{A^o} \to (E(A))'$ defined by $f(A^o) \mapsto f|E(A)$. This is seen to be an isometry of E'_{A^o} into $(E(A))'$. From this we have that the restriction of the transpose of I_B^A to E'_{B^o} is the surjective mapping $K_{A^o}^{B^o} : E'_{B^o} \to E'_{A^o}$.

Letting $E'' = (E'[\tau_b(E)])'$, the strong dual of E'_{A^o} is the space $E''(A^{oo})$ where A^o is a neighbourhood of 0 in $E'[\tau_b(E)]$; consequently A^{oo} is $\tau_b(E')$-bounded in E''. It can be shown that the mapping $H : E(A) \to E''(A^{oo})$ defined by $x \mapsto F_x$ where $F_x(f) = f(x)$ for each $f \in E'$, is an isometry into. Hence $E(A)$ is norm-isomorphic to a subspace of $(E'_{A^o})'$. It follows that the restriction of the transpose of $K_{A^o}^{B^o}$ to $E(A)$ is the injection $I_B^A : E(A) \to E(B)$.

13 <u>Definition and characterizations of Schwartz spaces</u>

This section begins with the original definition of a Schwartz space as given by GROTHENDIECK [7]. Various characterizations are given and a number of striking properties are then obtained.

4.8 DEFINITION A locally convex space $E[\tau]$ is said to be a *Schwartz space* if for every closed absolutely convex neighbourhood U of 0 in E, there exists a neighbourhood V of 0 such that for every $\alpha > 0$ the set V can be covered by finitely many translates of αU. #

The first theorem gives an immediate characterization of Schwartz spaces.

4.9 THEOREM A locally convex space $E[\tau]$ is a Schwartz space if and only if for every $U \in \mathcal{U}(E)$, the mapping $K_U : E \to E_U$ is precompact.

PROOF Let $E[\tau]$ be a Schwartz space, $U \in \mathcal{U}(E)$, and V be a neighbourhood of 0 with the property stated in 4.8. Let $\hat{x} = K_U(x)$ and $\hat{A} = K_U[A]$ for each $x \in E$ and $A \subset E$. Then for $\alpha > 0$, there is a finite subset $\{x_i\}_{i=1}^{n}$ of E such that

$$V \subset \bigcup_{i=1}^{n} (x_i + \alpha U). \quad \text{Hence } \hat{V} \subset \bigcup_{i=1}^{n} (\hat{x}_i + \alpha\hat{U}).$$ Because $\{\alpha\hat{U}\}_{\alpha>0}$ is a basis of neighbourhoods of 0 in E_U, the mapping K_U is precompact.

Suppose conversely K_U is precompact. Then there exists a neighbourhood V of 0 in E, such that for $\alpha > 0$, there exists a finite subset $\{x_i\}_{i=1}^{n}$ of points in E where $\hat{V} \subset \bigcup_{i=1}^{n} (\hat{x}_i + \frac{1}{2}\alpha\hat{U})$.

$$\text{Then } V \subset K_U^{-1}[\hat{V}] \subset K_U^{-1}\Big[\bigcup_{i=1}^{n} (\hat{x}_i + \tfrac{1}{2}\alpha\hat{U})\Big]$$

$$= \bigcup_{i=1}^{n} (x_i + \tfrac{1}{2}\alpha U + P) \quad \text{where } P = p_U^{-1}[\{0\}].$$

Since $P \subset \frac{1}{2}\alpha U$, $V \subset \bigcup_{i=1}^{n} (x_i + \alpha U)$. Hence $E[\tau]$ is a Schwartz space. #.

4.10 DEFINITION Let $E[\tau]$ be a locally convex space. A semi-norm p on E is said to be *precompact*, if the mapping $K_p : E \to E_p$ is precompact. #

As a corollary to 4.9, we have the following result. The proofs are obvious.

4.11 COROLLARY Let $E[\tau]$ be a locally convex space. Then the following statements are equivalent.

a) E is a Schwartz space;

b) for every $U \in \mathcal{U}(E)$, the mapping $K_U : E \to E_U$ is precompact;

c) for every $U \in \mathcal{U}(E)$, there exists $V \in \mathcal{U}(E)$, $V \subset U$ such that the mapping $K_U^V : E_V \to E_U$ is precompact;

d) for every $U \in \mathcal{U}(E)$, the mapping $\tilde{K}_U : E \to \tilde{E}_U$ is compact;

e) for every $U \in \mathcal{U}(E)$, there exists $V \in \mathcal{U}(E)$, $V \subset U$, such that the mapping $\tilde{K}_U^V : \tilde{E}_V \to \tilde{E}_U$ is compact; and

f) every continuous semi-norm on $E[\tau]$ is precompact. #

We now give a characterization of Schwartz spaces which is quite different from the previous ones.

4.12 THEOREM A locally convex space $E[\tau]$ is a Schwartz space if and only if the following two conditions are satisfied.

1) Every bounded subset of E is precompact; and

2) for every $U \in \mathcal{U}(E)$, there exists a neighbourhood V of 0 in E such that for every $\alpha > 0$, we can find a bounded set $A \subset E$ which satisfies $V \subset \alpha U + A$.

PROOF Let $U \in \mathcal{U}(E)$ and V be a neighbourhood of 0 as described in definition 4.8. If $B \subset E$ is bounded, then $B \subset \lambda V$ for some $\lambda > 0$. Let $\{x_i\}_{i=1}^n$ be a finite subset of E, such that $V \subset \bigcup_{i=1}^n (x_i + \frac{1}{\lambda} U)$. Then

$$B \subset \lambda V \subset \bigcup_{i=1}^n (\lambda x_i + U).$$ So B is precompact. Let $\alpha > 0$ and $\{y_i\}_{i=1}^n$ be a finite subset of E such that $V \subset \bigcup_{i=1}^n (y_i + \alpha U)$. Letting $A = \bigcup_{i=1}^n \{y_i\}$, then $V \subset \alpha U + A$, and A is obviously bounded.

Conversely, suppose conditions (1) and (2) are satisfied. Let $U \in \mathcal{U}(E)$, and V a neighbourhood of 0 which satisfies condition (2).

If $\alpha > 0$, let A be bounded such that $V \subset A + \frac{\alpha}{2} U$. Since A is precompact, there is a finite subset $\{x_i\}_{i=1}^{n}$ of E such that $A \subset \bigcup_{i=1}^{n} (x_i + \frac{\alpha}{2} U)$. Consequently, we have $V \subset [\bigcup_{i=1}^{n} (x_i + \frac{\alpha}{2} U)] + \frac{\alpha}{2} U$

$$= \bigcup_{i=1}^{n} (x_i + \alpha U).$$ So $E[\tau]$ is a Schwartz space. #

Using this result we now show that if $E[\tau]$ is a locally convex space, then $E[\tau_s(E')]$ is always a Schwartz space. It is a well-known fact that every $\tau_s(E')$-bounded set is $\tau_s(E')$-precompact.

A basic closed absolutely convex $\tau_s(E')$-neighbourhood of 0 in E is given by $U = \{x \in E : |\langle x, f_k\rangle| \le \varepsilon$ for $\varepsilon > 0$ and all $f_k \in E'$ where $1 \le k \le n\}$.

Suppose $\{f_k\}_{k=1}^{n}$ is not a linearly independent set. Then we have for some j, $f_j = \sum_{i=1}^{m} \alpha_i f_{k_i}$ (we may assume all $f_k(f_{k_i})$ are different and non-zero) where $f_j \neq f_{k_i}$, and $\alpha_i \neq 0$, for $i = 1, \ldots, m$.

Choosing ε' smaller than ε and each $\varepsilon/(m|\alpha_i|)$, $1 \le i \le m$, we have immediately

$$U' = \{x \in E : |\langle x, f_k\rangle| \le \varepsilon', k \neq j, 1 \le k \le n\} \subset U.$$

So we can assume the set $\{f_k\}_{k=1}^{n}$ is linearly independent.

Let $M \subset E'$ be the subspace generated by the set $\{f_k\}_{k=1}^{n}$. Define linear functionals F_i, $1 \le i \le n$, on M by

$$F_i\left(\sum_{k=1}^{n} \alpha_k f_k\right) = \alpha_i.$$ F_i for each i is $\tau_s(E)|M$-continuous since M is finite-dimensional and its topology is Hausdorff. So, for each i, there is a $\tau_s(E)$-continuous linear

functional G_i which extends F_i to E'.

Define $P : E' \to M$ by

$$f \mapsto \sum_{k=1}^{n} G_k(f) f_k. \quad \text{Clearly } P \text{ is}$$

continuous. If $f \in M$, $P(f) = \sum_{k=1}^{n} G_k(f)f_k = \sum_{k=1}^{n} F_k(f)f_k = f$.

For $f \in E'$,
$$\begin{aligned} P^2(f) &= \sum_{k=1}^{n} G_k\left(\sum_{i=1}^{n} G_i(f)f_i\right)f_k \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} G_k(f_i)G_i(f)f_k \\ &= \sum_{k=1}^{n} G_k(f)f_k = P(f). \end{aligned}$$

Hence P is a continuous projection of E' onto M. Let $N = P^{-1}[\{0\}]$. Then E' is the locally convex direct sum of M and N. Consequently, $I - P$, the projection of E' onto N is continuous.

Let $\{g_w\}_{w \in A}$ be a basis for N. Choose $V = U$, $A = \{x \in E : |\langle x, f_k\rangle| \le \varepsilon,\ 1 \le k \le n;$ and

$$|\langle x, g_w\rangle| \le \varepsilon,\ w \in A\}.$$ Any $f \in E'$ can be written as

$$f = \sum_{k=1}^{n} \alpha_k f_k + \sum_{w \in A} \beta_w g_w$$ for a suitable choice of scalars. So if $x \in A$, $|\langle x, f\rangle| \le \left(\sum_{k=1}^{n} |\alpha_k| + \sum_{w \in A} |\beta_w|\right)\varepsilon$, so A is $\tau_s(E')$-bounded in E.

Let F be any linear functional on E', such that $F[N] = \{0\}$. Then $F|M$ is $\tau_s(E)|M$-continuous since M is a finite-dimensional Hausdorff space. If we have a net $y_\alpha \to y$ in $E'[\tau_s(E)]$, then letting $m_\alpha + n_\alpha = y_\alpha$ for each α, and $m + n = y$, where m_α, $m \in M$ and n_α, $n \in N$, we obtain since the projection P is continuous that $m_\alpha \to m$.

Consequently $F(m_\alpha) \to F(m)$, and since $F[N] = \{0\}$, we have $F(y_\alpha) \to F(y)$; hence F is a $\tau_s(E)$-continuous linear functional.

Each linear functional $F' \in M^*$ may be extended to a $\tau_s(E)$-continuous linear functional F on E', such that $F[N] = \{0\}$. A basis for M^* is given by $\{F'_k\}_{k=1}^n$ where F'_k is defined by setting

$$F'_k \left(\sum_{i=1}^n \alpha_i f_i \right) = \alpha_k$$

. As before these obviously define continuous linear functionals on M.

The above mentioned extensions $\{F_k\}_{k=1}^n$ form a basis for the space $N^\perp \subset E$. Since each F_k is $\tau_s(E)$-continuous, there exists $x_k \in E$, such that

$$\langle x_k, f\rangle = F_k(f) \quad \text{for each } f \in E'.$$

If $x \in U$, set $u = \sum_{k=1}^n \langle x, f_k\rangle x_k$. Then for each f_k and g_w we have

$$|\langle u, f_k\rangle| = \left| \sum_{i=1}^n \langle x, f_i\rangle\langle x_i, f_k\rangle \right| = |\langle x, f_k\rangle| \leq \varepsilon$$

$$|\langle u, g_w\rangle| = \left| \sum_{i=1}^n \langle x, f_i\rangle\langle x_i, g_w\rangle \right| = 0, \quad \text{hence } u \in A.$$

Finally $\langle x - u, f_k\rangle = \langle x, f_k\rangle - \langle u, f_k\rangle = 0$.

Hence $x - u \in \alpha U$ for every $\alpha > 0$. So, $U \subset \alpha U + A$. Hence the result.

As a consequence, every finite-dimensional Hausdorff topological vector space is a Schwartz space. So we have

4.13 THEOREM A normed space is a Schwartz space if and only if it is finite-dimensional.

PROOF The sufficiency is clear from our remark. The necessity follows from 4.9 because by taking U to be the closed unit ball of a normed space $E[\tau]$, $E[\tau]'$ and E_U coincide as topological spaces. The result follows since a Hausdorff topological vector space has a precompact neighbourhood of 0 if and only if it is finite-dimensional. #

Next we characterize Schwartz spaces by means of a property of the dual spaces.

4.14 THEOREM Let $E[\tau]$ be a locally convex space with dual E'. Then E is a Schwartz space if and only if for every equicontinuous subset $B \subset E'$ there exists a neighbourhood V of 0 in E such that B is relatively compact in the normed space $E'(V^o)$.

PROOF Since B is equicontinuous $B \subset U^o$ for some $U \in \mathcal{U}(E)$. By corollary 4.11 there exists $V \in \mathcal{U}(E)$ such that $V \subset U$ and the mapping $K_U^V : E_V \to E_U$ is precompact. By 4.3 and statements in the previous section, the transpose of this mapping

$$I_{V^o}^{U^o} : E'(U^o) \to E'(V^o) \text{ is compact. So } U^o$$

and hence B is relatively compact in $E'(V^o)$.

Conversely if $U \in \mathcal{U}(E)$, then $B = U^o$ is equicontinuous. There exists $V \in \mathcal{U}(E)$ such that B is relatively compact in $E'(V^o)$. Obviously we may assume $V \subset U$. Hence the mapping

$$I_{V^o}^{U^o} : E'(U^o) \to E'(V^o) \text{ is compact. By 4.3 the mapping}$$

$K_U^V : E_V \to E_U$ is precompact. By corollary 4.11, $E[\tau]$ is a Schwartz space. #

TERZIOĞLU [19] and RANDTKE [14] gave the following characterization of

Schwartz spaces.

4.15 THEOREM If $E[\tau]$ is a locally convex space, $E[\tau]$ is a Schwartz space if and only if for every $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$, $V \subset U$, such that for every $\varepsilon > 0$ there exists a finite-dimensional subspace F of E such that

$$V \subset \varepsilon U + F.$$

PROOF By definition 4.8 the condition is necessary. So let $U \in \mathcal{U}(E)$ and V satisfy the above condition. Let $\varepsilon > 0$ and F a finite-dimensional subspace of E such that $V \subset \frac{\varepsilon}{2} U + F$.

Since V is absolutely convex and absorbing in E, $K_U[V \cap F]$ is absolutely convex and absorbing in $K_U[F] \subset E_U$. Because the topology of E_U restricted to the finite-dimensional subspace $K_U[F]$ is normable, we have $K_U[V \cap F]$ is a neighbourhood of 0 in $K_U[F]$; $K_U[U]$ is bounded in E_U, hence $K_U[U] \cap K_U[F]$ is bounded in $K_U[F]$, consequently $K_U[U \cap F] \subset K_U[U] \cap K_U[F]$ is bounded. So, there exists $\lambda > 0$ such that

$$\lambda K_U[U \cap F] \subset K_U[V \cap F].$$

Because $V \subset U$, $K_U[V \cap F] \subset K_U[U \cap F]$; thus $K_U[V \cap F]$ is bounded, and since $K_U[F]$ is finite-dimensional, $K_U[V \cap F]$ is totally bounded in $K_U[F]$ and hence in E_U.

So there is a finite subset $H \subset E$ such that

$K_U[V \cap F] \subset \lambda(1 + \frac{\varepsilon}{2})^{-1}(\frac{\varepsilon}{2})K_U[U] + K_U[H]$. If $x \in V$, there is a $y \in F$ such that

$x - y \in \frac{\varepsilon}{2} U$. From this and the fact that $V \subset U$, it follows that

$y \in (1 + \frac{\varepsilon}{2})(U \cap F)$. From above there is a $z \in V \cap F$ such that

$K_U(\lambda(1 + \frac{\varepsilon}{2})^{-1}y - z) = 0$. There exists $h \in H$ such that

$K_U(z - h) \in \lambda(1 + \frac{\varepsilon}{2})^{-1}(\frac{\varepsilon}{2})K_U[U]$. So now we have

$$K_U(x - \lambda^{-1}(1 + \frac{\varepsilon}{2})h) = K_U(x - y) + K_U(y - \lambda^{-1}(1 + \frac{\varepsilon}{2})z)$$
$$+ \lambda^{-1}(1 + \frac{\varepsilon}{2})K_U(z - h) \in \frac{\varepsilon}{2} K_U[U] + \frac{\varepsilon}{2} K_U[U]$$
$$= \varepsilon K_U[U].$$

So we have

$$\lambda K_U[V] \subset \varepsilon K_U[U] + \lambda^{-1}(1 + \frac{\varepsilon}{2}) K_U[H].$$

Since the sets $\{\varepsilon K_U[U]\}_{\varepsilon>0}$ form a basis of neighbourhoods of 0 for E_U, $K_U[V]$ is precompact, and K_U is a precompact mapping. By theorem 4.9, $E[\tau]$ is a Schwartz space. #

We now introduce the concept of local convergence.

4.16 DEFINITION A sequence $\{f_n\}_{n=1}^{\infty}$ in the dual E' of a locally convex space $E[\tau]$ *converges locally* if there exists a neighbourhood V of 0 in E such that each $f_n \in E'(V^o)$ and $\{f_n\}_{n=1}^{\infty}$ converges to an element of $E'(V^o)$ in the norm topology. #

Because $K_V : E[\tau] \to E_V$ is continuous, it follows from 4.1 that each locally convergent sequence is strongly convergent to the same limit.

4.17 THEOREM A subset A of a metrizable space $E[\tau]$ is precompact if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ converging to 0 in E such that each element of A can be written in the form $\sum_{n=1}^{\infty} \alpha_n x_n$ for some sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$.

If τ' is a finer locally convex topology on E such that $E[\tau']$ is sequentially complete, and if $\{x_n\}_{n=1}^{\infty}$ converges to 0 in $E[\tau']$, then each element of A has the same representation as before and A is precompact in $E[\tau']$.

PROOF The necessity of the first part is obtained from Lemma 2, Chapter VII of ROBERTSON and ROBERTSON [16]. The sufficiency is obvious since each point of A is contained in the closed absolutely convex hull of a null sequence.

So, assume A is precompact in the metrizable space $E[\tau]$, $x_n \to 0$ in $E[\tau']$, and each $x \in A$ can be written as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ in $E[\tau]$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars with $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$. For a closed absolutely convex neighbourhood U of 0 in $E[\tau']$, choose a positive integer N with $p_U(x_n) \leq 1$, for each $n \geq N$. If $m,n \geq N$, then $p_U(\alpha_n x_n + \ldots + \alpha_m x_m) \leq 1$. Consequently the finite partial sums $\{\sum_{n=1}^{k} \alpha_n x_n\}_{k=1}^{\infty}$ form a Cauchy sequence in $E[\tau']$. Since $E[\tau']$ is sequentially complete, $\sum_{n=1}^{\infty} \alpha_n x_n$ exists in $E[\tau']$. Because the sequence converges to x in a coarser topology, we must have $x = \sum_{n=1}^{\infty} \alpha_n x_n$ in $E[\tau']$. A is again precompact because it is contained in the closed absolutely convex hull of a null sequence. #

The next result typifies precompact semi-norms in terms of equicontinuous subsets of the dual and elements of c_o ([14]).

4.18 THEOREM A semi-norm p on a locally convex space $E[\tau]$ is precompact if and only if there is a sequence $\lambda \in c_o$ and an equicontinuous sequence $\{a_n\}_{n=1}^{\infty}$ in E' such that for each x in E

$$p(x) \leq \sup_n |\lambda_n| \, |\langle x, a_n \rangle| .$$

PROOF

Necessity: Letting $U = p^{-1}[[0, 1]]$, since p is precompact there is a $V \in \mathcal{U}(E)$ such that $V \subset U$, and $K_U^V : E_V \to E_U$ is precompact. By 4.3, its transpose

$$I_{V^o}^{U^o} : E'(U^o) \to E'(V^o)$$ is a compact mapping. So U^o is a precompact set in the Banach space $E'(V^o)$. By 4.17 there is a null sequence $\{b_n\}_{n=1}^{\infty}$ in $E'(V^o)$ such that U^o is contained in its closed absolutely convex hull in the form given by this result.

If p_{V^o} is the norm on $E'(V^o)$ with closed unit ball V^o, set $a_n = \frac{1}{p_{V^o}(b_n)} b_n$ if $b_n \neq 0$, and $a_n = 0$, if $b_n = 0$. For each n, set $\lambda_n = p_{V^o}(b_n)$.

Because $\{b_n\}_{n=1}^{\infty}$ is a null sequence, $p_{V^o}(b_n) \to 0$, hence $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in c_o$. Also for each n, either $p_{V^o}(a_n) = 0$ or $p_{V^o}(a_n) = 1$; so that each $a_n \in V^o$. So $\{a_n\}_{n=1}^{\infty}$ is an equicontinuous sequence.

Now $p(x) = \inf\{\lambda : x \in \lambda U\} = \sup\{f(x) : f \in U^o\}$. Because any sequence convergent in $E'(V^o)$ is weakly convergent in E', it can easily be seen that

$$p(x) \leq \sup_n |\langle x, b_n\rangle| = \sup_n |\lambda_n| \, |\langle x, a_n\rangle|.$$

Sufficiency: It is clear that any semi-norm on a locally convex space which is not larger than some precompact semi-norm at each point, is a precompact semi-norm. So, we need only show that

$$p(x) = \sup_n |\lambda_n| \, |\langle x, a_n\rangle|$$ is a precompact semi-norm on $E[\tau]$.

Because $\{a_n\}_{n=1}^{\infty}$ is an equicontinuous set, it is weakly bounded in E', and since $\lambda \in c_0$, p is a well-defined positive real-valued function on E. So, by properties of the supremum it is a semi-norm.

Define a linear function

$P : E \to \ell^\infty$ by $x \mapsto \{<x, a_n>\}_{n=1}^{\infty}$. We claim that P is a continuous function. Let a net $x_\alpha \to 0$ in E. For $\varepsilon > 0$, there is a neighbourhood U of 0 in E with $|<x, a_n>| \leq \varepsilon$ for each n and for each $x \in U$, because of the equicontinuity of $\{a_n\}_{n=1}^{\infty}$. Because there exists α_0 such that for $\alpha \geq \alpha_0$, we have $x_\alpha \in U$, then $||P(x_\alpha)||_\infty \leq \varepsilon$ for $\alpha \geq \alpha_0$. So $P(x_\alpha) \to 0$; so P is continuous.

Define $D_\lambda : \ell^\infty \to c_0$ by $\delta = \{\delta_n\}_{n=1}^{\infty} \mapsto \{\lambda_n\delta_n\}_{n=1}^{\infty}$. Obviously D_λ is well-defined and linear. We show that it is a precompact mapping. Let S be the unit ball in ℓ^∞. Choose $N > 0$ such that for $n > N$, $|\lambda_n| \leq \varepsilon/2$. Then, for each $s = \{s_n\}_{n=1}^{\infty} \in S$, $|\lambda_n s_n| \leq \varepsilon/2$ for $n > N$.

Restricting D_λ to the first N co-ordinates of each member of ℓ^∞, it is equivalent to a mapping

$$D_\lambda^N : \ell_N^\infty \to c_0 \quad \text{defined by}$$

$$\delta = \{\delta_n\}_{n=1}^{N} \mapsto \{w_n : w_n = \lambda_n\delta_n \text{ if } n \leq N;\ w_n = 0 \text{ if } n > N\}_{n=1}^{\infty}.$$

D_λ^N is linear, and hence continuous because ℓ_N^∞ is a finite-dimensional normed space. Restricting each point of S to its first N co-ordinates, we get the closed unit ball S_N of ℓ_N^∞, which is precompact in ℓ_N^∞. Consequently $D_\lambda^N[S_N]$ is precompact in c_0. So there is a finite subset F of c_0 with $D_\lambda^N[S_N] \subset F + S_{\varepsilon/2}$ where S_σ is the closed ball of radius σ in c_0. It is easily shown that $D_\lambda[S] \subset F + S_\varepsilon$. So D_λ is a precompact mapping.

Define a linear map $Q : D_\lambda[P[E]] \to E_p$ by $D_\lambda(P(x)) \mapsto K_p(x)$. If $D_\lambda(P(x)) = D_\lambda(P(y))$ then $D_\lambda(P(x - y)) = 0$, i.e. $\lambda_n \langle x - y, a_n \rangle = 0$ for each n. So $p(x - y) = 0$, hence $K_p(x - y) = 0$. Consequently the map Q is well-defined. Since the mapping Q is a norm-isomorphism, it is continuous.

Because $D_\lambda : \ell^\infty \to c_0$ is a precompact mapping, so is the restriction $D_\lambda : P[E] \to D_\lambda[P[E]]$. Also, $P : E \to P[E]$ is continuous. As a result, $K_p = QD_\lambda P : E \to E_p$ is a precompact mapping. So p is a precompact semi-norm on $E[\tau]$. #

The next corollary gives a characterization of precompact semi-norms in terms of locally convergent sequences.

4.19 COROLLARY A semi-norm p on a locally convex space $E[\tau]$ is precompact if and only if there is a sequence $\{f_n\}_{n=1}^\infty$ locally convergent to 0 in E', such that the inequality

$$p(x) \leq \sup_n |\langle x, f_n \rangle| \text{ holds for all } x \in E.$$

PROOF. The necessity as well as the sufficiency follow easily using the tools in the proof of theorem 4.18. #

As an immediate consequence of 4.11 and 4.18 we have the following theorem.

4.20 THEOREM. A locally convex space $E[\tau]$ is a Schwartz space if and only if for each continuous semi-norm p on E, there is a sequence $\lambda \in c_0$, and an equicontinuous sequence $\{a_n\}_{n=1}^\infty$ in E' such that for each x in E.

$$p(x) \leq \sup_n |\lambda_n| \, |\langle x, a_n \rangle| . \quad \#$$

The next result will be useful when we consider (DF)-spaces. We need a definition similar to the notion of a fundamental sequence of bounded sets.

4.21 DEFINITION A locally convex space $E[\tau]$ has a *fundamental sequence of precompact sets*, if there is a sequence $\{B_n\}$ of precompact sets of $E[\tau]$ such that $B_n \subset B_{n+1}$ for $n = 1, 2, \ldots,$ and if B is precompact in E, we have $B \subset B_n$ for some positive integer n. #

4.22 THEOREM If a locally convex space $E[\tau]$ has a fundamental sequence of precompact sets and if every $\tau_c(E)$-convergent sequence in E' is locally convergent, then E is a Schwartz space.

PROOF Because $\tau_c(E)$ and $\tau_s(E)$ coincide on τ-equicontinuous subsets of E', if U is a τ-neighbourhood of 0 in E, U^o is $\tau_c(E)$-compact. By hypothesis $\tau_c(E)$ is metrizable, so by theorem 4.17 we can find a sequence $\{f_n\}_{n=1}^{\infty} \subset E'$ such that $f_n \to 0$ in $\tau_c(E)$ and every element of U^o can be written as $\sum_{n=1}^{\infty} \alpha_n f_n$ for some scalar sequence $\{\alpha_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$. Again by hypothesis, we can find a τ-neighbourhood V of 0 in E such that $V \subset U$ and $f_n \to 0$ in the normed space $E'(V^o)$. By 4.17 U^o is precompact in $E'(V^o)$, and since U^o is closed in $E'(V^o)$, a Banach space, U^o is compact. By 4.14, the result follows. #

4.23 THEOREM If $E[\tau]$ is a Schwartz space and $U \in \mathcal{U}(E)$, then E_U is separable.

PROOF Let S be the closed unit ball of E_V where $V \in \mathcal{U}(E)$, $V \subset U$, and the mapping K_U^V is precompact.

Then $E_U = \bigcup_{n=1}^{\infty} K_U^V(nS) = \bigcup_{n=1}^{\infty} n K_U^V(S)$. Because $K_U^V(S)$ is precompact, $K_U^V(S)$ is contained in the closed absolutely convex hull of some null sequence $\{x_m\}_{m=1}^{\infty}$ in E_U.

Let $X_n = \bigcup_{m=1}^{\infty} \{nx_m\}$ for $n = 1, 2, \ldots$; then $X_n = nX_1$ and $\overline{\Gamma X_n} = n\overline{\Gamma X_1}$. Since $K_U^V(S) \subset \overline{\Gamma X_1}$, $n K_U^V(S) \subset n\,\overline{\Gamma X_1} = \overline{\Gamma X_n}$. So $E_U = \bigcup_{n=1}^{\infty} \overline{\Gamma X_n} = \overline{\Gamma \bigcup_{n=1}^{\infty} X_n}$.

Since each X_n is a countable set, $X = \bigcup_{n=1}^{\infty} X_n$ is a countable set and $E_U = \overline{\Gamma X}$. X can be written as the set $X = \{y_1, y_2, \ldots\}$. Let $Y_n = \{\sum_{i=1}^{n} \alpha_i y_i : \sum_{i=1}^{n} |\alpha_i| \leq 1$ where each α_i is rational (if $K = \mathbb{C}$, α_i has rational real and imaginary parts)$\}$; then Y_n is countable. It can be shown that $\Gamma X \subset \overline{Y} = \overline{\bigcup_{n=1}^{\infty} Y_n}$; consequently $E_U = \overline{Y}$; and thus E_U is separable. #

One obtains the following result.

4.24 COROLLARY Every metrizable Schwartz space is separable.

PROOF Let the sequence $\{U_n\}_{n=1}^{\infty}$ be a basis of absolutely convex closed neighbourhoods of 0 in a metrizable Schwartz space $E[\tau]$ such that $U_n + U_n \subset U_{n-1}$ for $n \geq 2$; let $K_{U_n} = K_n$; and for each $x \in E$ let $\hat{x}_n = K_n(x)$.

By the previous result, there exists for each n a sequence of points of E, $\{x_{nm}\}_{m=1}^{\infty}$, such that the sequence

$$\{(\hat{x}_{nm})_n\}_{m=1}^{\infty} \text{ is dense in } E_{U_n} = E_n.$$

Put $X = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{x_{nm}\}$; X is a countable subset of E. Let $x \in E$; then for each n, $K_{n+1}[x + U_{n+1}] = \hat{x}_{n+1} + K_{n+1}[U_{n+1}]$, and $(\hat{x}_{n+1} + K_{n+1}[U_{n+1}]) \cap K_{n+1}[X] \neq \phi$.

Hence $x + u_n - y = u$ for some $u_n \in U_{n+1}$, $y \in X$, and $u \in \bigcap_{\lambda>0} \lambda U_{n+1}$; consequently, $x - y \in U_{n+1} + U_{n+1} \subset U_n$. So, $(x + U_n) \cap X \neq \phi$. We have X is a countable dense subset of $E[\tau]$. #

14. Hereditary properties of Schwartz spaces

In this section, the hereditary properties of Schwartz spaces are examined. It is proved that linear subspaces, topological products, and quotient spaces by closed linear subspaces of Schwartz spaces are again Schwartz spaces. If there is a dense Schwartz subspace of a locally convex space, the original space is shown to be Schwartz. Fréchet-Schwartz spaces are defined, and conditions are given under which a locally convex space satisfies the hypotheses of 4.4 and 4.6.

4.25 THEOREM

a) A subspace M of a Schwartz space $E[\tau]$ is a Schwartz space.

b) Let $\{E_\alpha\}_{\alpha \in A}$ be a family of Schwartz spaces. Then the topological product $\prod_{\alpha \in A} E_\alpha$ is a Schwartz space.

c) Let $E[\tau]$ be a locally convex space. If M is a dense Schwartz subspace of $E[\tau]$, then $E[\tau]$ is a Schwartz space.

PROOF Let U be an absolutely convex closed neighbourhood of 0 in M of part (a). Then there exists an absolutely convex closed neighbourhood U_1 of 0 in $E[\tau]$ with $U_1 \cap M \subset U$. Let V_1 be a neighbourhood of 0 in $E[\tau]$ which is covered by finitely many translates of αU_1 for each

$\alpha > 0$. Take $V = V_1 \cap M$. Given $\alpha > 0$, there is a finite family $\{x_i\}_{i=1}^{n} \subset E$ such that $V_1 \subset \bigcup_{i=1}^{n} (x_i + \frac{\alpha}{2} U_1)$. Let H be the set of indices $1 \leq i \leq n$ with $(x_i + \frac{\alpha}{2} U_1) \cap M \neq \phi$. Then for $i \in H$ choose $y_i \in (x_i + \frac{\alpha}{2} U_1) \cap M$. Consequently $V \subset \bigcup_{i \in H} (y_i + \alpha U)$. So property (a) is proved.

Let U be an absolutely convex closed neighbourhood of 0 in $E = \prod_{\alpha \in A} E_\alpha$. Then there is a finite subset $H \subset A$, and an absolutely convex closed neighbourhood U_α of 0 in E_α for each $\alpha \in A$, such that $U_\alpha = E_\alpha$ for $\alpha \notin H$, and $\prod_{\alpha \in A} U_\alpha \subset U$. For each $\alpha \in H$ choose a neighbourhood V_α of 0 in E_α such that for every $\varepsilon > 0$, V_α is covered by finitely many translates of εU_α. For $\alpha \notin H$, choose $V_\alpha = E_\alpha$. Define $V = \prod_{\alpha \in A} V_\alpha$.

Given $\varepsilon > 0$, for every $\alpha \in H$, there exists a finite family $\{x_{\alpha k}\}_{k=1}^{n(\alpha)}$ of points in E_α, such that

$V_\alpha \subset \bigcup_{k=1}^{n(\alpha)} (x_{\alpha k} + \varepsilon U_\alpha)$. Denote by $\{y_j\}_{j=1}^{n}$, the family of points $y_j = \{y_{j\alpha}\}_{\alpha \in A}$ of E, where for $\alpha \in H$, $y_{j\alpha}$ equals some $x_{\alpha k}$, $1 \leq k \leq n(\alpha)$, and $y_{j\alpha} = 0$ for $\alpha \notin H$. Then $V \subset \bigcup_{j=1}^{n} (y_j + \varepsilon U)$. Hence, E is a Schwartz space.

Let U be an absolutely convex closed neighbourhood of 0 in the locally convex space $E[\tau]$. Then $U \cap M$ is an absolutely convex closed neighbourhood of 0 in M. So there exists V_1 neighbourhood of 0 in M, which for every $\alpha > 0$, is covered by finitely many translates of $\alpha(U \cap M)$. Hence, for $\alpha > 0$, we have a finite family $\{x_i\}_{i=1}^{n} \subset M$ such that

$V_1 \subset \bigcup_{i=1}^{n} (x_i + \alpha(U \cap M))$. Setting $V = \overline{V_1}$, the closure of V_1 in $E[\tau]$, $V \subset \bigcup_{i=1}^{n} (x_i + \alpha U)$, since the latter is a closed set. Because V is a neighbourhood of 0 in $E[\tau]$, $E[\tau]$ is a Schwartz space. #

4.26 DEFINITION A locally convex space $E[\tau]$ is called a *Fréchet-Schwartz space* (an (FS)-space) if it is both an (F)-space and a Schwartz space. #

Because infrabarrelled and quasi-complete Schwartz spaces are (M)-spaces, we have the following corollary.

4.27 COROLLARY Every (FS)-space is an (FM)-space. #

We now show that the class of Schwartz spaces is closed under taking quotient spaces by closed linear subspaces.

4.28 THEOREM Let $E[\tau]$ be a Schwartz space and M a closed subspace. Then the quotient space $E/M\ [\tau_q]$ is a Schwartz space.

As a consequence of this, if $E[\tau]$ is an (FS)-space, $E/M\ [\tau_q]$ is an (FS)-space.

PROOF Let $\hat{U}$ be an absolutely convex closed neighbourhood of 0 in E/M. Then if $K : E \to E/M$ is the canonical surjection, $K^{-1}[\hat{U}] = U$ is an absolutely convex closed neighbourhood of 0 in $E[\tau]$. Choose a neighbourhood V of 0 in $E[\tau]$ which satisfies the definition of a Schwartz space. Let $\hat{V} = K[V]$.

If $\alpha > 0$, there is a finite set $\{x_i\}_{i=1}^{n} \subset E$ with $V \subset \bigcup_{i=1}^{n} (x_i + \alpha U)$. Then setting $K(x_i) = \hat{x}_i$, $1 \leq i \leq n$, we have

$$\hat{V} \subset \bigcup_{i=1}^{n} (\hat{x}_i + \alpha\hat{U}).$$ So $E/M\ [\tau_q]$ is a Schwartz space.

The consequence follows from HORVATH [9, Ch. 2, Th. 9.2]. #

In Chapter II we gave an example of a quotient space of an (FM)-space, by a closed subspace, which was not an (M)-space. As a consequence of the previous theorem, not every (FM)-space is a Schwartz space.

We now show (FS)-spaces satisfy the conditions of 4.4 and 4.6.

4.29 THEOREM Let $E[\tau]$ be an (FS)-space and $M \subset E$ be a closed subspace. Then,

a) the strong topology on $E'/M^{\perp}$, i.e. $\tau_b(M)$, coincides with the quotient topology $\tau_b(E)_q$ on $E'/M^{\perp}$, and

b) the strong topology on E', $\tau_b(E)$, relativized to $M^{\perp}$, coincides with the strong topology $\tau_b(E/M)$.

PROOF Because $E[\tau]$ is an (M)-space result (a) follows from 4.6 since (M)-spaces are reflexive.

If B is bounded in $E/M\,[\tau_q]$, then its closed absolutely convex hull is compact. By 4.28, $E/M[\tau_q]$ is an (F)-space; so by 4.7 there exists $H \subset E$ which is absolutely convex and compact, such that $K[H] = \overline{\Gamma B}$. By 4.4, result (b) follows. #

The next result is useful in considering strict inductive limits of sequences.

4.30 THEOREM Let E be a vector space and let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of linear subspaces of E such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n \in \mathbb{N}} E_n$. Suppose each E_n is equipped with a locally convex topology τ_n for which $E_n[\tau_n]$ is a Schwartz space. Suppose, further, that $\tau_{n+1}|E_n = \tau_n$, and E_n is closed in E_{n+1} for τ_{n+1}, for each $n \in \mathbb{N}$.

Let τ be the finest locally convex topology (not necessarily Hausdorff) for which the injections $I_n : E_n \to E$ are continuous. Then $E[\tau]$ is a

Schwartz space.

PROOF. By [9; Ch. 2, Cor. 1 of Th. 12.1], τ is Hausdorff, so that $E[\tau]$ is the strict inductive limit of the spaces $\{E_n[\tau_n]\}_{n \in \mathbb{N}}$, and so by [9; Ch. 2, Th. 12.1], $\tau | E_n = \tau_n$ for each $n \in \mathbb{N}$.

We shall verify that the conditions of 4.12 are satisfied.

If $B \subset E[\tau]$ is bounded, then by the Dieudonné-Schwartz Theorem [9; Ch. 2, Th. 12.2], $B \subset E_n$ for some n, and is τ_n-bounded there. Since $E_n[\tau_n]$ is a Schwartz space, B is τ_n-precompact; consequently B is precompact in $E[\tau]$.

Let U be an absolutely convex closed neighbourhood of 0 in $E[\tau]$. Then $U_n = I_n^{-1}[U] = U \cap E_n$ is an absolutely convex closed neighbourhood of 0 in $E_n[\tau_n]$; let V_n be a neighbourhood of 0 in $E_n[\tau_n]$ with $V_n \subset U_n$, and V_n satisfying the second condition of 4.12. Let $V = \Gamma_{n=1}^{\infty} \frac{1}{n} V_n$. Then V is a neighbourhood of 0 in $E[\tau]$ since $V \cap E_n \supset \frac{1}{n} V_n$ is a neighbourhood of 0 in $E_n[\tau_n]$ for each n.

Let $\alpha > 0$. Then we have

$$\frac{1}{n} V_n \subset \alpha U_n \subset \alpha U \quad \text{for} \quad n \geq 1/\alpha .$$

For $1 \leq n \leq \frac{1}{\alpha}$ let A_n be bounded in $E_n[\tau_n]$ such that $\frac{1}{n} V_n \subset A_n + \alpha U_n$. Let A be the absolutely convex hull of the union of the sets A_n, $1 \leq n \leq 1/\alpha$, and $A = \{0\}$, if no $n \leq 1/\alpha$.

Then A is bounded in $E[\tau]$ and $\frac{1}{n} V_n \subset A + \alpha U$ for $n \in \mathbb{N}$. Since $A + \alpha U$ is absolutely convex, $V \subset A + \alpha U$. #

CHAPTER V

CO-SCHWARTZ AND UNIVERSAL SCHWARTZ SPACES

In this chapter we introduce the class of co-Schwartz spaces as a sort of dual to the situation of Schwartz spaces. As in Schwartz spaces it is shown that bounded sets are precompact. We then examine the classes of (F)-spaces which are Schwartz or co-Schwartz and (DF)-spaces which are Schwartz or co-Schwartz.

In [5] it was announced that there exists a "universal" Schwartz space E, meaning that every Schwartz space can be obtained as a subspace of some power of E, but no concrete example was available. In this chapter, concrete examples of three such universal Schwartz spaces are exhibited, namely c_0, ℓ_0^∞, and $C[0,1]$ with certain topologies. This result is due to RANDTKE [15].

§5 Co-Schwartz spaces and their applications to (DF)-spaces

First, we give the definition of co-Schwartz spaces given by TERZIOGLU [19].

5.1 DEFINITION A locally convex space $E[\tau]$ is a *co-Schwartz space* if for every $A \in \mathcal{B}(E)$, there exists another set $B \in \mathcal{B}(E)$ such that $A \subset B$ and the injection $I_B^A : E(A) \to E(B)$ is precompact. #

5.2 DEFINITION

a) An (F)-space which is a co-Schwartz space is called an *(FcS)-space*.

b) A (DF)-space which is also a Schwartz space is called a *(DFS)-space*.

c) A (DF)-space which is a co-Schwartz space is called a *(DFcS)-space*. #

We now give a justification for the nomenclature of a co-Schwartz space.

5.3 THEOREM A locally convex space $E[\tau]$ is a co-Schwartz space if and only if its strong dual $E'[\tau_b(E)]$ is a Schwartz space.

PROOF The sets A^o where $A \in \mathcal{B}(E)$ form a basis of closed absolutely convex neighbourhoods of 0 for $E'[\tau_b(E)]$. For any $A \in \mathcal{B}(E)$, there exists $B \in \mathcal{B}(E)$ with $A \subset B$, and I_B^A precompact. By 4.3 the transpose of I_B^A is compact, hence $K_{A^o}^{B^o}$, the restriction of the transpose to E'_{B^o} is precompact. By 4.11 $E'[\tau_b(E)]$ is a Schwartz space.

If $E'[\tau_b(E)]$ is a Schwartz space, then for every bounded $A \in \mathcal{B}(E)$, there exists bounded $B \in \mathcal{B}(E)$ such that $A \subset B$, and the mapping $K_{A^o}^{B^o} : E'_{B^o} \to E'_{A^o}$ is precompact. So the restriction of the transpose of $K_{A^o}^{B^o}$ to $E(A)$, which is the injection I_B^A, is also precompact. Hence $E[\tau]$ is a co-Schwartz space. #

The next lemma will be needed in showing that every (DFcS)-space is infrabarrelled.

5.4 LEMMA Every bounded subset of a co-Schwartz space $E[\tau]$ is precompact and separable.

PROOF. If $A \subset E$ is τ-bounded, then $A \subset B_1$, for some $B_1 \in \mathcal{B}(E)$. So there exists $B_2 \in \mathcal{B}(E)$ such that B_1, and hence A, is a precompact subset of the normed space $E(B_2)$. By a method similar to the proof of 4.23 it can be shown A is a subset of some separable subset of the normed space $E(B_2)$. Since a metrizable topological space is separable if and only if it has a countable basis of open sets, A is separable as a subspace of $E(B_2)$. Because the injection $I : E(B_2) \to E[\tau]$ is continuous, A is also separable and precompact in the space $E[\tau]$. #

We now apply the theory of Schwartz and co-Schwartz spaces to (F)- and (DF)-spaces. First, we give a condition for an (F)-space to be co-Schwartz.

5.5 THEOREM An (F)-space $E[\tau]$ is an (FcS)-space if and only if every bounded subset of E is precompact.

PROOF The necessity was proved in 5.4; so we prove the sufficiency. Let $A \in \mathcal{B}(E)$. By assumption, and 4.17, there exists a null sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that every element of A can be written as $\sum_{n=1}^{\infty} \alpha_n x_n$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars with $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$.

Let $C = \Gamma_{n=1}^{\infty} \{x_n\}$. Then C is an absolutely convex bounded set. For each integer $n \geq 1$, choose $\lambda_n \geq n$ such that $C \subset \frac{\lambda_n}{n} U_n$. Let $B = \bigcap_{n=1}^{\infty} \lambda_n U_n$ (where we have taken the sequence $\{U_n\}_{n=1}^{\infty}$ to be a basis of absolutely convex closed neighbourhoods of 0 in $E[\tau]$). Then B is closed bounded and absolutely convex in $E[\tau]$.

It can easily be shown that if p_B is the norm associated with the normed space $E(B)$, then $p_B(x_n) \to 0$. Since $E[\tau]$ is an (F)-space, B is sequentially complete as a subspace of $E[\tau]$; so by [13; 20, 11.(2)], $E(B)$ is a Banach space. By 4.17, A is precompact in the Banach space $E(B)$, and $A \subset B$, so $E[\tau]$ is an (FcS)-space. #

As an immediate corollary we have the following.

5.6 COROLLARY An (F)-space is an (FcS)-space if and only if it is an (FM)-space. #

By [13; 27, 2.(5)] every (FM)-space is separable; so by 5.6, every (FcS)-space is separable. We proved in 4.24 that every (FS)-space is separable. In the previous section, we noted the existence of an (FM)-

space which is not a Schwartz space. Consequently, not every (FcS)-space is an (FS)-space. But as another corollary to 5.5, we have the next result.

5.7 COROLLARY Every (FS)-space is an (FcS)-space. #

We now study the dual situation of (DF)-spaces. Since a (DF)-space has a fundamental sequence of bounded sets, we have the following result which comes from 5.4.

5.8 THEOREM Every (DFcS)-space is separable, and hence infrabarrelled. #

The next result deals with the topology of a (DFS)-space.

5.9 LEMMA If $E[\tau]$ is a (DFS)-space, then the topology τ of E is the topology of uniform convergence on the relatively $\tau_b(E)$-compact subsets of E'.

PROOF We show that the classes of τ-equicontinuous and relatively $\tau_b(E)$-compact subsets of E' coincide. Since the bounded and precompact subsets of E coincide, the topologies $\tau_b(E)$ and $\tau_c(E)$ coincide. On τ-equicontinuous subsets of E', $\tau_c(E)$ and $\tau_s(E)$ coincide; consequently every equicontinuous subset of E' is relatively $\tau_b(E)$-compact.

Conversely, if A is a relatively compact subset of the (F)-space $E'[\tau_b(E)]$, there exists a null sequence $\{f_n\}_{n=1}^{\infty} \subset E'$ such that A is contained in the $\tau_b(E)$-closed absolutely convex hull of the sequence. Since $E[\tau]$ is a (DF)-space, the sequence $\{f_n\}_{n=1}^{\infty}$ is equicontinuous because it is strongly bounded. We can find a neighbourhood U of 0 in $E[t]$ with $\{f_n\}_{n=1}^{\infty} \subset U^o$, which is $\tau_b(E)$-closed and absolutely convex. Then $A \subset U^o$ and the proof is complete. #

We now have the following result about strong duals of (DFS)-spaces.

5.10 THEOREM The strong dual of a (DFS)-space $E[\tau]$ is an (FM)-space.

PROOF We have proved in chapter II that $E'[\tau_b(E)]$ is an (F)-space, so we need only show every $\tau_b(E)$-bounded set is relatively $\tau_b(E)$-compact. This is equivalent to showing every closed and bounded subset A of $E'[\tau_b(E)]$ is compact. Since the topology $\tau_b(E)$ restricted to A is metrizable, we need only show that every sequence in A has a convergent subsequence. This is satisfied by showing every strongly bounded sequence of E' has a convergent subsequence. This is easy since by 5.9 every strongly bounded sequence is equicontinuous, and, consequently, relatively $\tau_b(E)$-compact. #

If $E[\tau]$ is a (DFS)-space every strongly bounded subset of E' is relatively $\tau_b(E)$-compact, by 5.10, and by 5.9, is equicontinuous. As for (DFcS)-spaces, we have the next result.

5.11 THEOREM Every (DFS)-space is infrabarrelled. #

5.12 LEMMA If $E[\tau]$ is a (DF)-space, every bounded subset of E is precompact, and every strongly convergent sequence in E' is locally convergent, then $E[\tau]$ is a (DFS)-space.

PROOF By assumption we have $\tau_c(E) = \tau_b(E)$; consequently, every $\tau_c(E)$-convergent sequence is locally convergent. By theorem 4.22 $E[\tau]$ is a Schwartz space. #

Using this lemma we show that

5.13 THEOREM A (DFM)-space $E[\tau]$ is a (DFS)-space.

PROOF. Since $E[\tau]$ is an (M)-space, the bounded and relatively compact sets coincide, thus so do the bounded and precompact sets.

Let $u_n \to u$ strongly in E'. As in the proof of 5.5, with slight alterations, there is a closed absolutely convex bounded set B of $E'[\tau_b(E)]$ such that $p_B(u_n - u) \to 0$ and u and each u_n are all in B. Because (M)-spaces are infrabarrelled $\bar{B}$ is equicontinuous; so $B \subset U^o$ where U is some τ-neighbourhood of 0 in E. Because $B \subset U^o$, we have that $p_{U^o}(u_n - u) \to 0$. Hence $u_n \to u$ locally. By lemma 5.12 $E[\tau]$ is a (DFS)-space. #

Let $E[\tau]$ be an (FM)-space which is not a Schwartz space. Then $E'[\tau_b(E)]$ is a (DFS)-space by 5.13, observing that the strong dual of an (M)-space is an (M)-space by [13; 27, 2.(2)]. The strong dual of $E'[\tau_b(E)]$ is $E[\tau]$ because (M)-spaces are reflexive; consequently $E'[\tau_b(E)]$ is not a co-Schwartz space.

In contrast we now have a result dual to statement 5.7.

5.14 THEOREM. A (DFcS)-space $E[\tau]$ is a (DFS)-space.

PROOF $E'[\tau_b(E)]$ is an (FM)-space. The bidual $E''[\tau_b(E')]$ is a (DFM)-space. By 5.13 the bidual is a Schwartz space. By lemma 5.8, $E[\tau]$ is infrabarrelled and, thus is a subspace of the bidual. By theorem 4.25, $E[\tau]$ is a Schwartz space. #

To conclude the section a condition is given under which (DFS)- and (DFcS)-spaces are (M)-spaces.

5.15 THEOREM If $E[\tau]$ is a quasi-complete (DFS)-space or a quasi-complete (DFcS)-space, then $E[\tau]$ is a (DFM)-space.

PROOF. Since (DFcS)-spaces are (DFS)-spaces we need only consider quasi-complete (DFS)-spaces. By theorem 5.11, $E[\tau]$ is infrabarrelled. As noted earlier, infrabarrelled and quasi-complete Schwartz spaces are (M)-spaces. #

16 Universal Schwartz spaces

We need a few preliminary results before we reach the main result.

5.16 LEMMA Let $E[\tau]$ be a metrizable locally convex space, and $A \subset E$ be precompact. Then there is an absolutely convex closed bounded subset B of E such that $A \subset B$ and A is precompact in $E(B)$.

PROOF Because A is precompact there is a null sequence $\{x_n\}_{n=1}^{\infty}$ in $E[\tau]$ such that $A \subset \overline{\Gamma_{n=1}^{\infty} \{x_n\}} = C$. If $\tilde{E}[\tilde{\tau}]$ is the completion of $E[\tau]$, then $\tilde{E}[\tilde{\tau}]$ is a metrizable space and $x_n \to 0$ in $\tilde{E}[\tilde{\tau}]$. So A is contained in the closed absolutely convex hull C' of $\{x_n\}_{n=1}^{\infty}$ in $\tilde{E}[\tilde{\tau}]$, which is the closure of C in $\tilde{E}[\tilde{\tau}]$.

As in 5.5 we can find an absolutely convex closed bounded subset B' of $\tilde{E}$ such that $C' \subset B'$ and $p_{B'}(x_n) \to 0$ where $p_{B'}$ is the norm defined on $\tilde{E}(B')$ with closed unit ball B'. Considering the restriction of $\tilde{\tau}$ to $\tilde{E}(B')$, we have the closed absolutely convex hull of $\{x_n\}_{n=1}^{\infty}$ in $\tilde{\tau}|\tilde{E}(B')$ is C' since B' contains C'. Hence A is precompact in the restricted topology of $\tilde{\tau}$ to $\tilde{E}(B')$.

By [13; 20, 11.(2)] $\tilde{E}(B')$ is a Banach space, so by theorem 4.17, A is precompact in $\tilde{E}(B')$. Let $B = B' \cap E$; then $\tilde{E}(B') \cap E(B) = E(B)$. If $x \in E(B)$, then $x \in \lambda B'$ if and only if $x \in \lambda(B' \cap E) = \lambda B$; so $p_{B'}(x) = p_B(x)$. Consequently, $E(B)$ is a subspace of $\tilde{E}(B')$. Since $A \subset B$, A is precompact in $E(B)$. #

The next result is about precompact semi-norms.

5.17 LEMMA Let $E[\tau]$ be a locally convex space. If p and q are precompact semi-norms on E, then $p + q$ is a precompact semi-norm on E. Let $F[\tau']$ be another locally convex space, q a continuous semi-norm on F, and $T : E \to F$ a continuous linear map. If either q or T is precompact, then qT is a precompact semi-norm on E.

PROOF The space $E_p \times E_q$ is a product of normed spaces, and can easily be shown to be normed by [cf. page 75]

$||(K_p(x),\ K_q(y))|| = p(x) + q(y)$. Define a linear map $T' : E \to E_p \times E_q$ by $x \mapsto (K_p(x),\ K_q(x))$. Then the kernel of T' is the set $(p + q)^{-1}[\{0\}]$. So the linear map $S : E_{(p+q)} \to T'[E]$ defined by $K_{p+q}(x) \mapsto T'(x)$ is a well-defined bijection.

Obviously S is an isometric isomorphism of $E_{(p+q)}$ onto $T'[E]$. Because p and q are precompact, T' can be shown to be precompact. Restricting the range of T' to $T'[E]$, then $T' = SK_{p+q}$; hence K_{p+q} is a precompact mapping.

We now prove the latter part of the statement. It is easily shown that qT is a continuous semi-norm on E. We shall show E_{qT} is isometrically isomorphic to $K_q[T[E]]$. The former is normed by $||K_{qT}(x)|| = qT(x)$, while the latter is normed by $||K_q(T(x))|| = qT(x)$. So the mapping $S : E_{qT} \to K_q[T[E]]$ defined by $K_{qT}(x) \mapsto K_q(T(x))$ is an isometric isomorphism. Then $K_{qT} = S^{-1}K_qT$ if we restrict the map K_q to $K_q : T[E] \to K_q[T[E]]$ and the map T to $T : E \to T[E]$. The result now follows easily. #

5.18 DEFINITION Let $E[\tau]$ and $F[\tau']$ be locally convex spaces and $T : E \to F$ a linear map. T is said to be a *quasi-Schwartz map* if there is a precompact semi-norm p on E such that the set $\{T(x) : p(x) \leq 1\}$ is a bounded set in F. #

This definition will enable us to give a necessary and sufficient condition for a map between normed spaces to be precompact.

5.19 LEMMA Let $E[\tau]$, $F[\tau']$ and $G[\tau'']$ be locally convex spaces. Let $T : E \to F$ and $S : F \to G$ be continuous linear maps.

If either T or S is quasi-Schwartz, then ST is quasi-Schwartz.

PROOF Suppose T is quasi-Schwartz and p is a precompact semi-norm on E such that $\{T(x) : p(x) \leq 1\}$ is bounded in F. Then $\{ST(x) : p(x) \leq 1\} = S[\{T(x) : p(x) \leq 1\}]$ and the continuous linear image of a bounded ~~set~~ is bounded.

If S is quasi-Schwartz and q is a precompact semi-norm on F with $\{S(y) : q(y) \leq 1\}$ bounded in G, then by 5.17 qT is a precompact semi-norm on E. We also have

$$\{ST(x) : qT(x) \leq 1\} \subset \{S(y) : q(y) \leq 1\} . \quad \#$$

The next result shows the equivalence of precompact and quasi-Schwartz linear maps into metrizable spaces.

5.20 THEOREM Let $E[\tau]$ and $F[\tau']$ be locally convex spaces and $T : E \to F$ a linear map.

If T is quasi-Schwartz, then T is precompact.

If F is metrizable, and T is precompact, then T is quasi-Schwartz.

PROOF Suppose T is quasi-Schwartz. There exists a precompact semi-norm p on E, such that $M = \{T(x) : p(x) \leq 1\}$ is bounded in F. Since $T[p^{-1}[\{0\}]] \subset M$, and is a subspace, $T[p^{-1}[\{0\}]] = \{0\}$. Then, the linear map $S : E_p \to F$ defined by $K_p(x) \mapsto T(x)$ is well defined.

Because M is bounded, S is continuous; since K_p is precompact, we have $T = SK_p$ is precompact.

Let T be precompact and $F[\tau']$ metrizable. There is a continuous semi-norm q on E with $M' = \{T(x) : q(x) \leq 1\}$ a precompact subset of F. Then $M = \overline{M'}$ is also precompact. By 5.16 there is a set $N \in \mathcal{B}(F)$ such that the injection

$$I_N^M : F(M) \to F(N) \text{ is precompact, and } M \subset N.$$

The restricted mapping $T : E \to F(M)$ is continuous, so the restricted mapping $T : E \to F(N)$ is precompact. Let q' be the semi-norm of $F(N)$ identified with the closed unit ball N. By lemma 5.17, $p = q'T$ is a precompact semi-norm on E. The set $\{T(x) : p(x) \leq 1\} \subset N$ and so is bounded. Consequently the map $T : E \to F$ is quasi-Schwartz. #

5.21 COROLLARY A linear map $T : E \to F$ from a locally convex space $E[\tau]$ into a normed space $F[||\cdot||]$ is precompact if and only if there is a precompact semi-norm p on E such that for each $x \in E$,

$$||T(x)|| \leq p(x).$$

PROOF First, assume T is precompact. By 5.20, T is quasi-Schwartz, so there is a precompact semi-norm q on E, such that there exists an integer $N > 0$, for which if $x \in E$ and $q(x) \leq 1$, then $||T(x)|| \leq N$. Choose $p = Nq$; then by 5.17, p is a precompact semi-norm on E. It is easy to see that $||T(x)|| \leq p(x)$ for each $x \in E$.

Conversely define a semi-norm q on E by $q(x) = ||T(x)||$ for each $x \in E$. Then $q \leq p$, so q is precompact. The set $\{T(x) : q(x) \leq 1\} = \{T(x) : ||T(x)|| \leq 1\}$ is norm-bounded in F; consequently, T is quasi-Schwartz and hence precompact. #

Suppose $E[\tau]$ is a Schwartz space, and $T : E \to F$ is a continuous linear map into a normed space $F[|| \cdot ||]$. Then there is a $U \in \mathcal{U}(E)$ such that $T[U]$ is normed-bounded by 1 in F.

The mapping $S : E_U \to F$ defined by $K_U(x) \mapsto T(x)$ is thus well-defined and continuous. Because K_U is a precompact mapping, $T = SK_U$ is a precompact mapping. Consequently, we have the following result.

5.22 LEMMA Let $E[\tau]$ be a locally convex space. Then $E[\tau]$ is a Schwartz space if and only if every continuous linear map from E into a normed space is precompact, and, hence, quasi-Schwartz. #

The next result is due to RANDTKE [14].

5.23 THEOREM Let $E[\tau]$ be a locally convex space. The following statements are equivalent.

a) $E[\tau]$ is a Schwartz space;

b) every bounded linear map from E into a locally convex space $F[\tau']$ is quasi-Schwartz; and

c) every precompact linear map from E into a locally convex space $F[\tau']$ is quasi-Schwartz.

PROOF Suppose $E[\tau]$ is a Schwartz space. Let $M = \overline{T[U]}$ in F, where $T : E \to F[\tau']$ is a bounded linear map, $T[U]$ is bounded, and $U \in \mathcal{U}(E)$. Then $S : E \to F(M)$ defined by $x \mapsto T(x)$ is continuous. By 5.22, S is quasi-Schwartz. If $K : F(M) \to F$ is the canonical injection, then $T = KS$

is quasi-Schwartz by 5.19. So we have (a) implies (b).

Since every precompact map is bounded, it follows that (b) implies (c).

So let $F[||\cdot||]$ be a normed space and $T : E \to F$ be a continuous linear map. Then T is a bounded linear map. Because a bounded subset of F is $\tau_s(F')$-precompact, we have, by (c), that T is $\tau_s(F')$-quasi-Schwartz. So there is a precompact semi-norm p on E such that $\{T(x) : p(x) \leq 1\}$ is $\tau_s(F')$-bounded in F. But the norm-bounded and $\tau_s(F')$-bounded subsets of F coincide; consequently T is quasi-Schwartz, and hence precompact. By 5.22, we have (c) implies (a). #

We now establish the existence of a finest Schwartz space topology on a space coarser than a given locally convex topology.

5.24 THEOREM Let $E[\tau]$ be a locally convex space with dual E'. Then there is a Schwartz space topology τ' on E for which $(E[\tau'])' = E'$, τ' is coarser than τ, and τ' is finer than any Schwartz space topology on E which is coarser than τ.

PROOF Let τ' be the locally convex topology on E defined by the τ-precompact semi-norms. Because $\tau_s(E')$-continuous semi-norms are $\tau_s(E')$-precompact, hence τ-precompact, τ' is a locally convex topology admissible for the pairing $\langle E,E'\rangle$, and $\tau' \leq \tau$. If $\beta \leq \tau$ is a topology on E such that $E[\beta]$ is a Schwartz space, then any β-continuous semi-norm on E is β-precompact, hence τ-precompact. Obviously $\beta \leq \tau'$.

It remains to show that $E[\tau']$ is a Schwartz space. By lemma 5.17 multiples and sums of τ-precompact semi-norms are τ-precompact; so it suffices to show each τ-precompact semi-norm is τ'-precompact. We show that if $\lambda \in c_0$ and $\{a_n\}_{n=1}^{\infty}$ is a τ-equicontinuous sequence in E', then there exist $u \in c_o$ and $\{b_n\}_{n=1}^{\infty}$, a τ'-equicontinuous sequence of E' such that for each $x \in E$ we have

$$\sup_n |\lambda_n| \; |\langle x, a_n\rangle| \le \sup_n |u_n| \; |\langle x, b_n\rangle|$$

From 4.11 and 4.18, we see this is sufficient.

For λ, choose sequences u, $\sigma \in c_o$ such that $u_n\sigma_n = \lambda_n$ for each n. Set $b_n = \sigma_n a_n$ for each n.
Then for each $x \in E$,

$$|\langle x, b_n\rangle| = |\sigma_n| \; |\langle x, a_n\rangle| \le \sup_n |\sigma_n| \; |\langle x, a_n\rangle| .$$

So $\{b_n\}_{n=1}^{\infty}$ is a τ'-equicontinuous sequence in E'. And, obviously,
$\sup_n |\lambda_n| \; |\langle x, a_n\rangle| = \sup_n |u_n| \cdot |\langle x, b_n\rangle|$ for each $x \in E$. #

5.25 DEFINITION A Schwartz space E is *universal* if every Schwartz space is topologically isomorphic to a linear subspace of some product E^I of E.

Let I be a partially-ordered set and $\{E_u\}_{u \in I}$ a family of locally convex spaces. Suppose for each pair (u, v) of indices such that $u \le v$ we have a continuous linear map

$f_{uv} : E_v \to E_u$, such that f_{uu} is the identity map for each $u \in I$, and $f_{uw} = f_{uv} \circ f_{vw}$ for $u \le v \le w$. Let E be the topological subspace of the topological product $\prod_{u \in I} E_u$ formed by the vectors $\{x_u\}_{u \in I}$ satisfying $f_{uv}(x_v) = x_u$ for $u \le v$. The locally convex space E is called the *projective limit* of the *projective system* $\{E_u, f_{uv}\}_{u,v \in I}$. The projective limit of the projective system $\{E_u, f_{uv}\}_{u,v \in I}$ where the E_u are Banach spaces is said to be *compact* if for each u, there exists $v \ge u$ such that the map $f_{uv} : E_v \to E_u$ is compact. In a paper, "A Structure Theorem for Schwartz Spaces" Math. Ann. 201(1973), 171-176, RANDTKE proved the following result characterizing Schwartz spaces. We note that if $E_u = E$ for each u, the projective limit is said to be a projective limit of E-spaces.

5.26 THEOREM If E denotes any one of the Banach spaces c_o, ℓ^∞ or $C[0,1]$, then a locally convex space $E[\tau]$ is a Schwartz space if and only if it is topologically isomorphic to a linear subspace of a compact projective limit of E-spaces. #

Using this result we give concrete examples of universal Schwartz spaces.

5.27 THEOREM Let E denote any one of the spaces c_o, ℓ^∞ or $C[0,1]$. If β denotes the topology on E defined by the precompact semi-norms on E, then $E[\beta]$ is a universal Schwartz space.

PROOF By theorem 5.24, $E[\beta]$ is a Schwartz space. Let $F[\tau]$ be a Schwartz space; then by 5.26, $F[\tau]$ is topologically isomorphic to a linear subspace of the compact projective limit P of a system $\{E, f_{uv}\}_{u,v \in I}$.

Let T denote the natural injection of P into E^I. Let S denote the identity map from E^I onto $E[\beta]^I$. Since β is coarser than the Banach space topology on E, S is continuous. We show that $ST : P \to E[\beta]^I$ is relatively open.

Let $u \in I$, and U denote the set of all points x in P with $||x_u|| \leq 1$. Choose $v \geq u$ in I with $f_{uv} : E \to E$ compact. By 5.21, let p be a precompact semi-norm on E such that

$$||f_{uv}(x)|| \leq p(x) \quad \text{for each } x \in E.$$

Let V denote the set of x in $E[\beta]^I$ such that $p(x_v) \leq 1$. Then, if $ST(x) \in V \cap ST[P]$, we have $p((ST(x))_v) \leq 1$. So $||f_{uv}((ST(x))_v)|| = ||(ST(x))_u|| \leq 1$. Consequently $ST(x) \in ST[U]$; so ST is relatively open.

So, $F[\tau]$ is topologically isomorphic to a linear subspace of $E[\beta]^I$. #

BIBLIOGRAPHY

1. ADASCH, N. : Über die vollständigkeit von $L_\sigma(E,F)$, Math. Ann. 191 (1971), 290-292.

2. BOURBAKI, N. : Éléments de mathématique, Livre V, Espaces vectoriels topologiques, 2 Bände, Hermann, Paris, Nr. 1189, 1229 (1953, 1955).

3. COOPER, J.B. : The strict topology and spaces with mixed topologies, Proc. Amer. Math. Soc. 30,3 (1971), 583-592.

4. DIEUDONNÉ, J. : Sur les propriétés de permanence de certains espaces vectoriels topologiques, Ann. Soc. Pol. Math. 25, (1952), 50-55.

5. DIESTEL, J.; MORRIS, S.A.; and SAXON, S.A. : Varieties of locally convex linear topological spaces, Bull. Amer. Math. Soc. 77 (1971), 799-803.

6. ERNST, B. : Ultra-(DF)-räume, J. Reine und Angew. Math. 258 (1973), 87-102.

7. GROTHENDIECK, A. : Sur les espaces (F) et (DF), Summa Brasil. Math. 3, (1954), 57-123.

8. " : Espaces vectoriels topologiques, Departmento de Matemática da Universidade de São Paulo (1954).

9. HORVÁTH, J. : Topological vector spaces and distributions, Vol. I, Addison-Wesley, Reading, Mass. (1966).

10. IYAHEN, S.O. : On certain classes of linear topological spaces, Proc. London Math. Soc. 3, 18 (1968), 285-307.

11. KŌMURA, Y. : On linear topological spaces, Kumamoto J. Science, Series A, 5, Nr. 3 (1962), 148-157.

12. KŌMURA, T. and KŌMURA, Y. : Sur les espaces parfaits de suites et leurs généralisations, J. Math. Soc. Japan 15 (1963), 319-338.

13. KÖTHE, G. : Topological vector spaces I, Springer-Verlag, Berlin (1969).

14. RANDTKE, D.J. : Characterizations of precompact maps, Schwartz spaces and nuclear spaces, Trans. Amer. Math. Soc. 165 (1972), 87-101.

15. " : A simple example of a universal Schwartz space, Proc. Amer. Math. Soc. 37, 1 (1973), 185-188.

16. ROBERTSON, A.P. and ROBERTSON, W.J. : Topological vector spaces, Cambridge Mathematical Tracts, No. 53 (1964).

17. SCHAEFER, H.H. : Topological vector spaces, Springer-Verlag (1971).

18. SCHWARTZ, L. : Functional analysis, New York University, Courant Institute of Mathematical Sciences, New York (1964).

19. TERZIOǦLU, T. : On Schwartz spaces, Math. Ann. 182 (1969), 236-242.

20. VALDIVIA, M. : Absolutely convex sets in barrelled spaces, Ann. Inst. Fourier, Grenoble, 21, 2 (1971), 3-13.

21. " : On (DF)-spaces, Math. Ann. 191 (1971), 38-43.

22. VALDIVIA, M. : A class of quasi-barrelled (DF)-spaces which are not bornological, Math. Z. 136 (1974), 249-251.